Unifying colored sl(2) link homology

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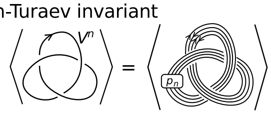
Abstract
In the recent project with Anna Beliakova, StephanWehrli and Matt Hogancamp we are comparing two different categorications of the colored Jones polynomial: Khovanov's construction (based on a resolution of a symmetric representation in the Temperly-Lieb category) and Cooper-Krushkals' (via categorified Jones-Wenzl projectors). Khovanov's construction is finite-dimensional, but not defined in a purely diagrammatic way. On the other hand, Cooper-Krushal's complex can be constructed diagrammatically, but the homology has innite dimension. In this talk I will discuss their quantized versions and how we show that they coincide.

Colored Jones polynomial

The *n*-colored Jones polynomial of a framed oriented knot K is the Reshetikhin-Turaev invariant

$J_n(K) = P_{sl(2)}(K, V^n) =$ 

where $p_n \in TL_{n,n}$ is the Jones-Wenzl projector that corresponds to

$$V^{\otimes n} \twoheadrightarrow V^n \hookrightarrow V^{\otimes n}$$

fund. rep.

It is uniquely determined by
❶ p_n = id$_n$ + (flat tangles of width < n)
❷ p_n kills turnbacks

Categorified colored Jones polynomial

To categorify the colored Jones polynomial one has to find a collection of complexes

$$C_{sl(2)}(K, V^n)$$

whose graded Euler characteristic is $J_n(K)$.

Better: a complex for each representation such that
❶ $C_{sl(2)}(K, V)$ is the Khovanov's complex
❷ ~~$C_{sl(2)}(K, W\oplus W') = C_{sl(2)}(K, W) \oplus C_{sl(2)}(K, W')$~~

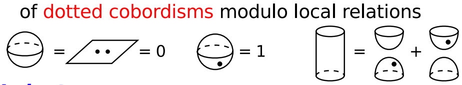
There is a filtration on $C_{sl(2)}(K, W\oplus W')$ with $\mathrm{gr}(C_{sl(2)}(K, W\oplus W')) = C_{sl(2)}(K, W) \oplus C_{sl(2)}(K, W')$

Ex: $C_{sl(2)}(K, V^n\otimes V)$ vs $C_{sl(2)}(K, V^{n-1}) \oplus C_{sl(2)}(K, V^{n+1})$

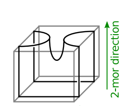
Bar-Natan (bi)category *BN*

The natural home for Khovanov's invariant

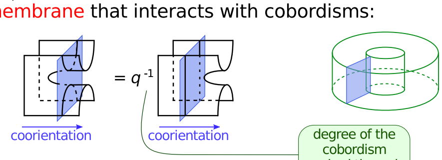
- objects: (sequences of) planar diagrams
- morphisms: (matrices of) linear combinations of dotted cobordisms modulo local relations



Variants
❶ Extension down (for tangles):
obj's: points on a line
mor's: flat tangles
2-mor's: codordisms with corners



❷ $BN(A)$: diagrams on an annulus, cobordisms in a thickened annulus
❸ $BN_q(A)$: as above but with a (cooriented) vertical membrane that interacts with cobordisms:

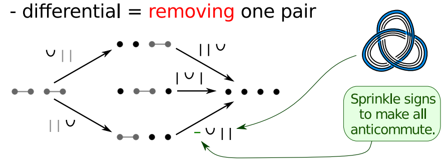


Theorem (*BPW*) For generic q:
$$\mathrm{hTr}_q(BN) \simeq BN_q(A) \simeq TL \text{ (via } S^1 \times (\text{--})\text{)}$$
Idea: apply the annular closure in the directon of 1-morphisms.
The hard part: show that the functor is faithful.

References
D.Bar-Natan: arXiv:math/0410495
A.Beliakova, K.Putyra & S.Wehrli: arXiv:1605.03523
B.Cooper & S.Krushkal: arXiv:1005.5117
M.Hogancamp: arXiv:1405.2574
M.Khovanov: arXiv:math/0302060

Khovanov's colored homology

❶ Construct a resolution $\mathcal{V}^{(n)} \in Com(TL)$ of V^n:
- start with *n* dots on a line
- *k*-chains = *k* disjoint pairs of neighbored dots
- differential = removing one pair



❷ Reinterprete (unmatched) dots as (the Khovanov bracket of) a cabling of K and the differential as an annulus cobordism.

Problem
Khovanov bracket is functorial only up to homotopy.
The above lives in $Com(K^b(TL))$

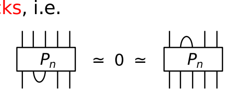
htpy category of complexes

Solutions
❶ Take the actual homology instead of the bracket.
This is the original approach by Khovanov.
❷ Find higher differentials (homotopies, homotopies between homotopies, etc.)
Figured out by Stephan Wehrli
❸ Consider knots as twisted traces on the Bar-Natan bicategory. This leads to satellite operators
$$K_*\colon BN(A) \longrightarrow BN(A)$$
This is the approach we took (work in progress)

Cooper-Krushkal homology

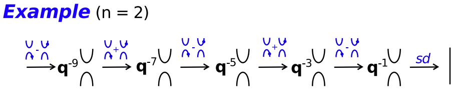
aka categorified Jones-Wenzl projectors

There is a unique complex $P_n \in K^-(BN_{n,n})$ of the form $P_n = Cone(T_n \longrightarrow \mathrm{id}_n)$ with T_n spanned by flat tangles of width < n (T_n is called the tail of P_n) and which kills turnbacks, i.e.



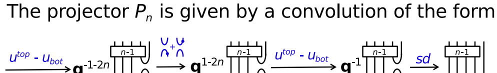
Properties
❶ $P_n \circ P_n \simeq P_n$
❷ P_n categorifies the Jones-Wenzl projector
❸ T_n is a (2*n*-2)-periodic complex
❹ There is a unique (up to scalar) endomorphism $u_n\colon P_n \longrightarrow \mathbf{q}^{2n-2}P_n$ (placing a dot for *n*=2)

Example (*n* = 2)



When closed in an annulus:
$$\xrightarrow{0} \mathbf{q}^{-9}A \xrightarrow{X} \mathbf{q}^{-7}A \xrightarrow{0} \mathbf{q}^{-5}A \xrightarrow{X} \mathbf{q}^{-3}A \xrightarrow{0} \mathbf{q}^{-1}A \xrightarrow{rk\,1} A^{\otimes 2}$$
whereas in a quantized annulus:
$$\xrightarrow{(1-q^2)X} \mathbf{q}^{-9}A \xrightarrow{(1+q^2)X} \mathbf{q}^{-7}A \xrightarrow{(1-q^2)X} \mathbf{q}^{-5}A \xrightarrow{(1+q^2)X} \mathbf{q}^{-3}A \xrightarrow{(1-q^2)X} \mathbf{q}^{-1}A \xrightarrow{rk\,1} A^{\otimes 2}$$
acyclic!

Theorem (*BHPW*)
The projector P_n is given by a convolution of the form



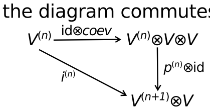
Homological models for colors

Represent symmetric colors by complexes

Fix a monoidal category ($\mathcal{C}$, ⊗, 1) and a self-dual object $V \in \mathcal{C}$ (the fundamental color).

A homological model for *sl*(2) colors consists of
- complexes $V^{(n)} \in \mathrm{Com}^-(\mathcal{C})$ with $V^{(0)} = 1$ and $V^{(1)} = V$,
- chain maps
$p^{(n)}\colon V^{(n)}\otimes V \longrightarrow V^{(n+1)}$
$i^{(n)}\colon V^{(n)} \longrightarrow V^{(n+1)}\otimes V$

such for n≤0 the diagram commutes up to homotopy



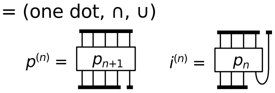
and there is a distinguished triangle
$$V^{(n)} \xrightarrow{i^{(n)}} V^{(n+1)}\otimes V \xrightarrow{p^{(n+1)}} V^{(n+2)} \longrightarrow V^{(n)}[1]$$
in the homotopy category $K^-(\mathcal{C})$.

Properties
❶ $V^{(n+2)} \simeq Cone(i^{(n)})$
❷ All homological models are homotopy equivalent.
❸ There is a dual model, with all arrows reversed. It appears that all models are self dual (even when $\mathcal{C}$ has no duals!)

Note A homological model for colors is a colored homology for the unknot.

Jones-Wenzl model

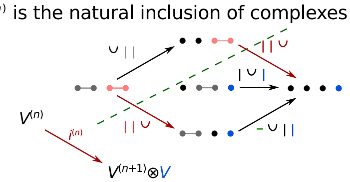
❶ ($\mathcal{C}$, ⊗, 1) = (*Kar*(*TL*), ⊔, (empty))
im(p_n) is represented by a platform
❷ (V, *ev*, *coev*) = (one dot, ∩, ∪)
❸

$V^{(n)}$ = im(p_n) 

Theorem
- im(p_{n+2}) ≃ Cone($i^{(n)}$)
- $p^{(n+1)}$ is the canonical inclusion

Khovanov model

❶ ($\mathcal{C}$, ⊗, 1) = (*TL*, ⊔, (empty))
❷ (V, *ev*, *coev*) = (one dot, ∩, ∪)
❸ $p^{(n)}$ is the natural inclusion of complexes



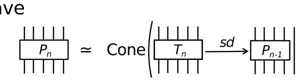
Cooper-Krushkal model

Here we need quantum annulus!

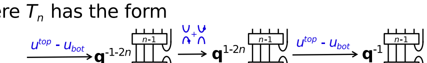
Notation: [P_n] = the quantum annular closure on P_n

Theorem (BHPW)
The quantum annular closures of the projectors P_n constitute a homological model for sl(2) colors.

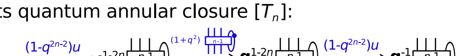
Sketch of the proof
We have



where T_n has the form



Its quantum annular closure [T_n]:



Expanding the projectors once more we get