A TOUR OF SKEIN MODULES

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Knots on Web (ICTS Bengaluru)
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What is a skein module?

Alexander polynomial (1928)

Alexander - Conway polynomial (1969)

Alexander - Conway polynomial (1969) Jones polynomial (1984)

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HOMFLYPT polynomial (1984)

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Vassiliev - Gusarov invariant (1989)

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Kauffman bracket polynomial (1985)

Dubrovnik polynomial (1985)

Vassiliev - Gusarov invariant (1989)

Invariants of links in S^3 and Skein Relations

- Skein modules are generalisations of the various polynomial link invariants in S^3 to arbitrary 3-manifolds.
- They are 3-manifold invariants.
- They were introduced by Józef H. Przytycki in 1987 (independently by Vladimir Turaev in 1988)
- There are many different skein modules!

Why are skein modules important?

Skein Modules

Skein Modules

Algebraic Geometry

Hyperbolic Geometry

Skein Modules

Algebraic Geometry

Hyperbolic Geometry

Skein Modules

Witten - Reshetikhin - Turaev Invariants

Algebraic Geometry

Hyperbolic Geometry

Topological Quantum Field Theories

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Witten - Reshetikhin - Turaev Invariants

Representation Theory

Homological invariants

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Quantum Cluster algebras

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Topological Quantum Field Theories

Skein Modules

Hecke algebras

Hopf algebras

Witten - Reshetikhin - Turaev Invariants

Representation Theory

Homological invariants

OUTLINE

- Examples of skein modules and results
- The Kauffman bracket skein module
- Properties of Kauffman bracket skein modules
- Kauffman bracket skein modules of some 3-manifolds
- Witten's finiteness conjecture
- Kauffman bracket skein algebras
- Kauffman bracket skein algebras of some 3-manifolds
- The positivity conjecture
- Properties of Kauffman bracket skein algebras
- Connection to the $SL(2,\mathbb{C})$ character variety

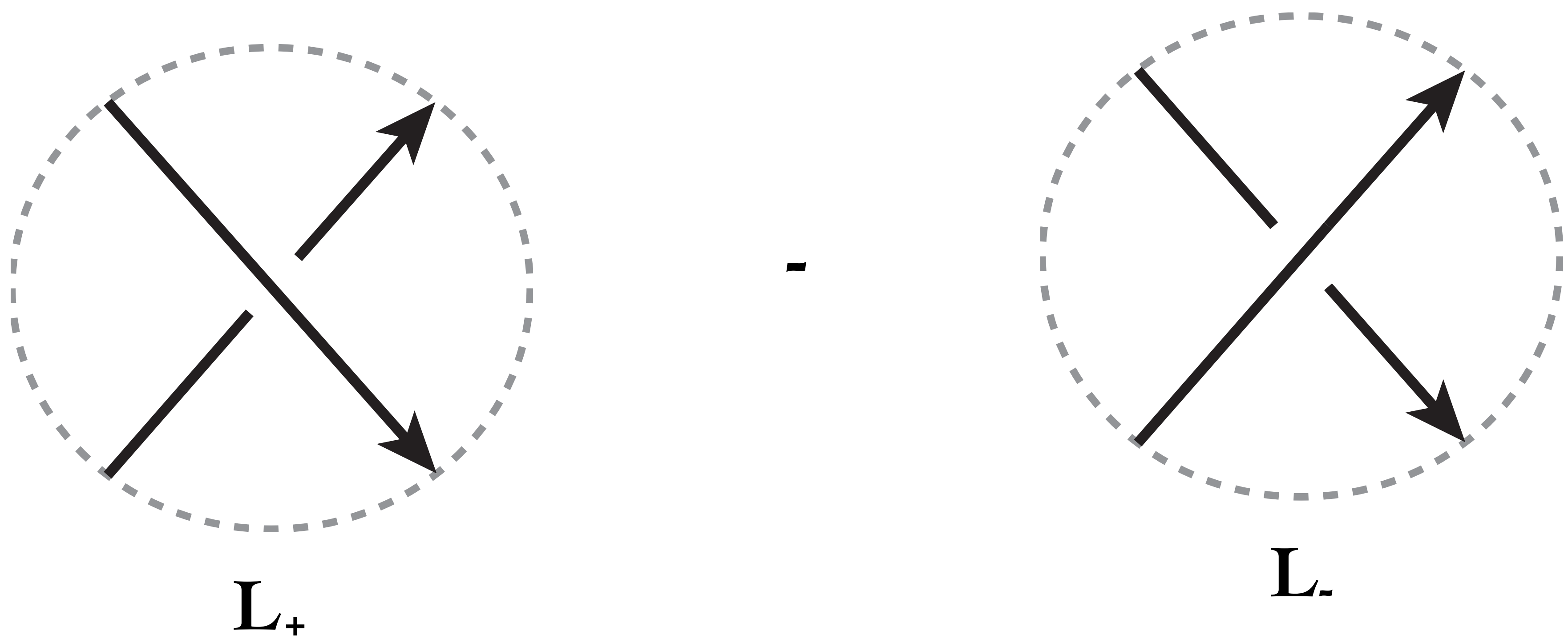
Notation

- M^3: oriented 3-manifold
- $\overrightarrow{\mathscr{L}}$: set of ambient isotopy classes of oriented links
- $\mathscr{L}^{fr}$: set of ambient isotopy classes of unoriented framed links
- $\mathscr{L}^{\overrightarrow{fr}}$: set of ambient isotopy classes of oriented framed links
- These may have the empty link $\emptyset$
- R : a commutative ring with unity and/or a fixed invertible element A or q
- $F_{g,b}$: oriented genus g surface with b boundary components

The Signed Skein Module

The Signed Skein Module

Let $\overrightarrow{S}$ be the submodule of $R\overrightarrow{\mathscr{L}}$ generated by the following skein expression:



Skein Relation

$$\overrightarrow{\mathscr{S}}_{\pm}(M^3; R, L_+ - L_-) = \overrightarrow{\mathscr{S}}_{\pm}(M^3) = R\overrightarrow{\mathscr{L}} / \overrightarrow{S}$$

$$\overrightarrow{\mathcal{S}}_{\pm}(M^3; R, L_+ - L_-) = \overrightarrow{\mathcal{S}}_{\pm}(M^3) = R\overrightarrow{\mathcal{L}}/\overrightarrow{S}$$

- $\overrightarrow{\mathcal{S}}_{\pm}(M^3)$ is a free module over the homotopy classes of links in M^3.

- $\overrightarrow{\mathcal{S}}_{\pm}(M^3)$ admits a natural algebra structure: $L_1 \cdot L_2 = L_1 \sqcup L_2$ and $\emptyset$ is the multiplicative identity.

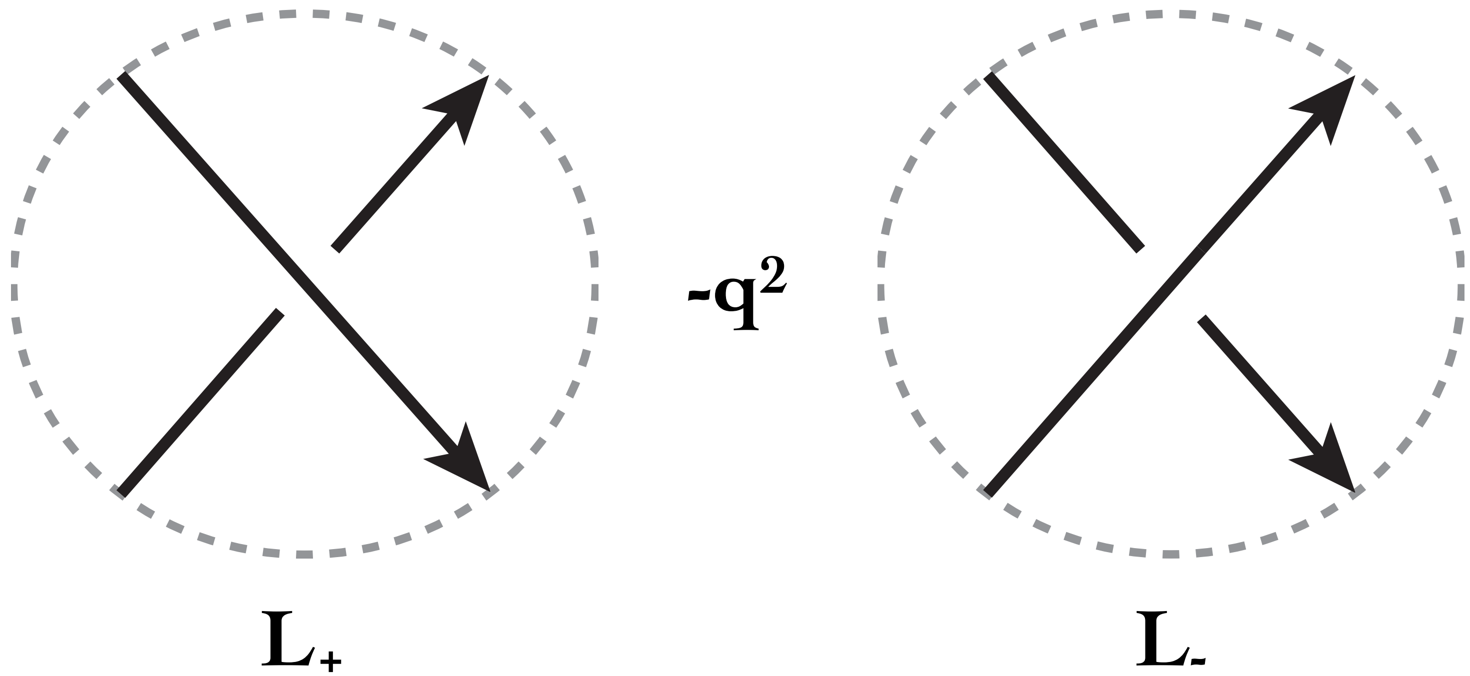
- $\overrightarrow{\mathcal{S}}^A_{\pm}(M^3)$ is a commutative algebra isomorphic to the polynomial algebra with coefficients in R and the set of variables is the set of conjugacy classes of $\pi_1(M^3)$.

Framing Version of the Signed Skein Module

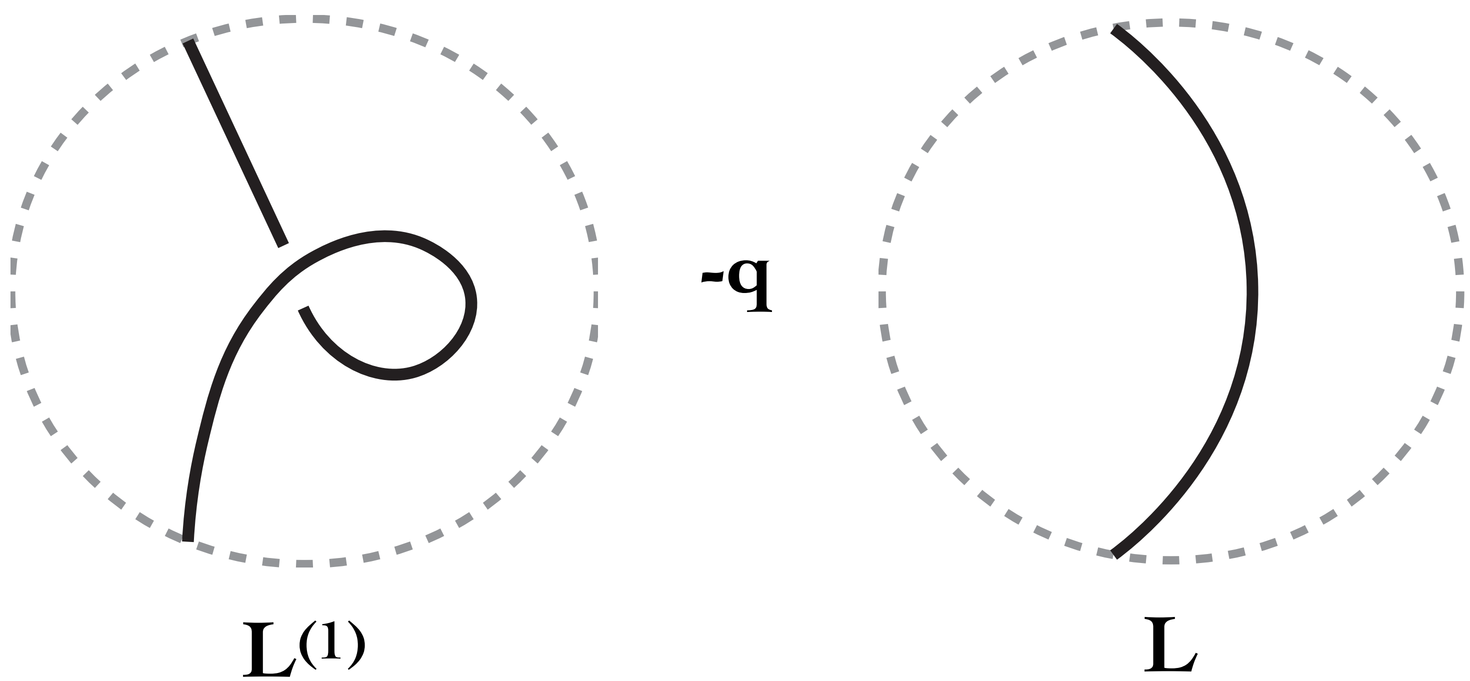
Framing Version of the Signed Skein Module

Let $R = \mathbb{Z}[q^{\pm 1}]$ and $\overrightarrow{S^{fr}}$ be the submodule of $R\overrightarrow{\mathscr{L}^{fr}}$ generated by the following skein expressions:

Skein Relation:



Framing Relation:



$$\mathcal{S}_{\pm}^{\overrightarrow{fr}}(M^3) = R\mathscr{L}^{\overrightarrow{fr}}/S^{\overrightarrow{fr}}$$

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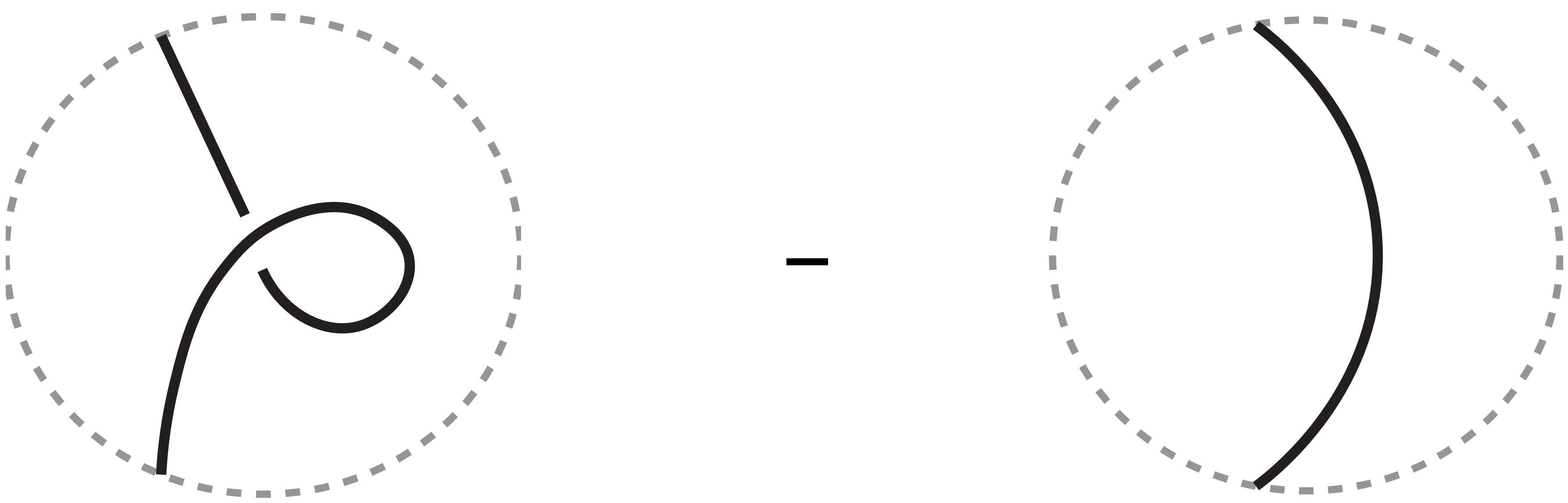
Przytycki (1997) showed that if M^3 contains a non-separating 2-sphere or torus then $\mathcal{S}_{\pm}^{\overrightarrow{fr}}(M^3)$ has torsion and Kaiser (1997) showed that torsion occurs only in these cases.

The Framing Skein Module

The Framing Skein Module

Let S^{fr} be the submodule of the module $\mathbb{Z}[q^{\pm 1}]\mathscr{L}^{fr}$ generated by:



L$^{(1)}$ – **L**

Framing Relation

$$\mathcal{S}_0(M^3, q) = \mathbb{Z}[q^{\pm 1}]\mathscr{L}^{fr}/S^{fr}$$

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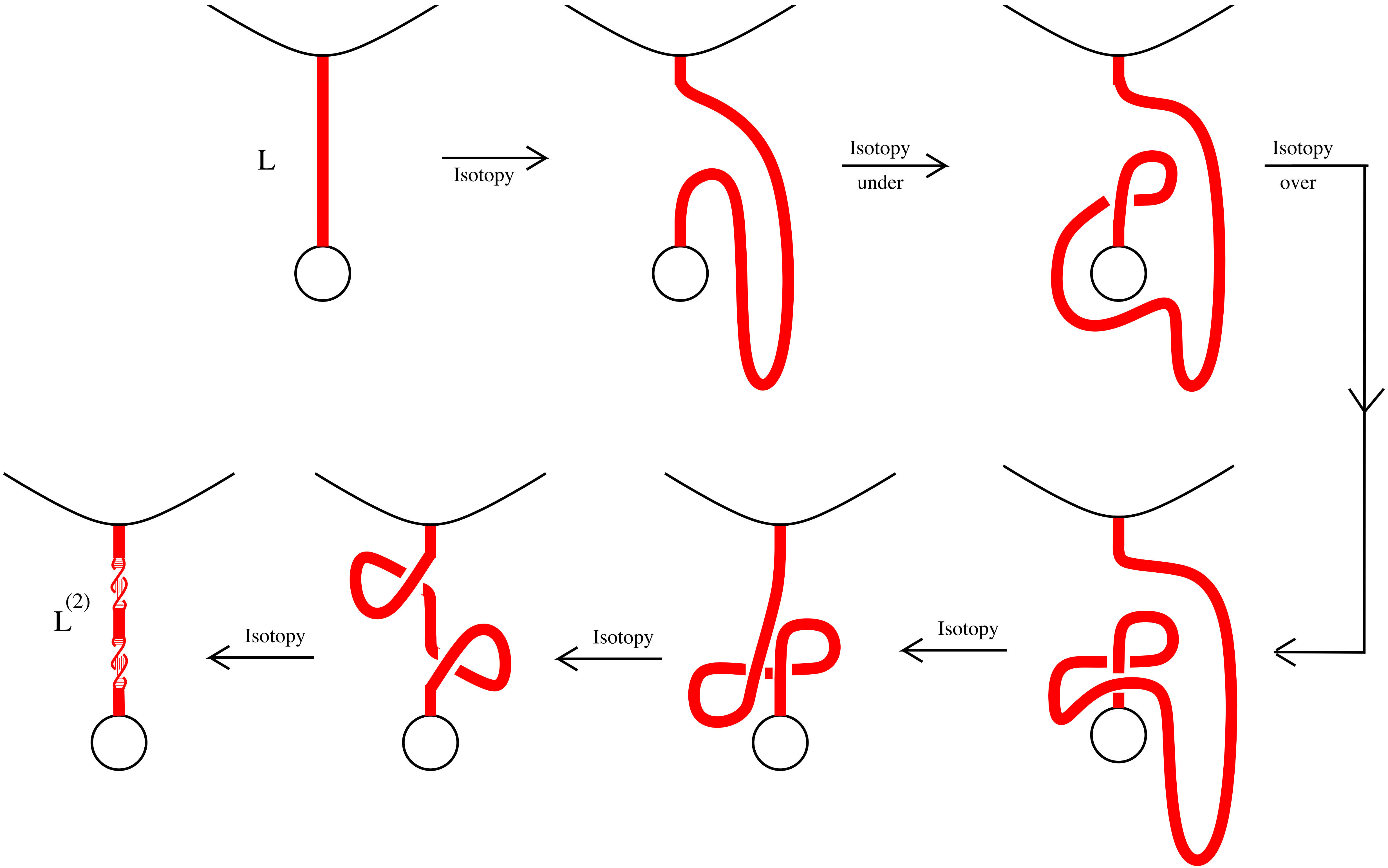
The framing skein module detects the presence of non-separating 2-spheres in M^3.

$$\mathcal{S}_0(M^3, q) = \mathbb{Z}[q^{\pm 1}]\mathcal{L}^{fr}/S^{fr}$$

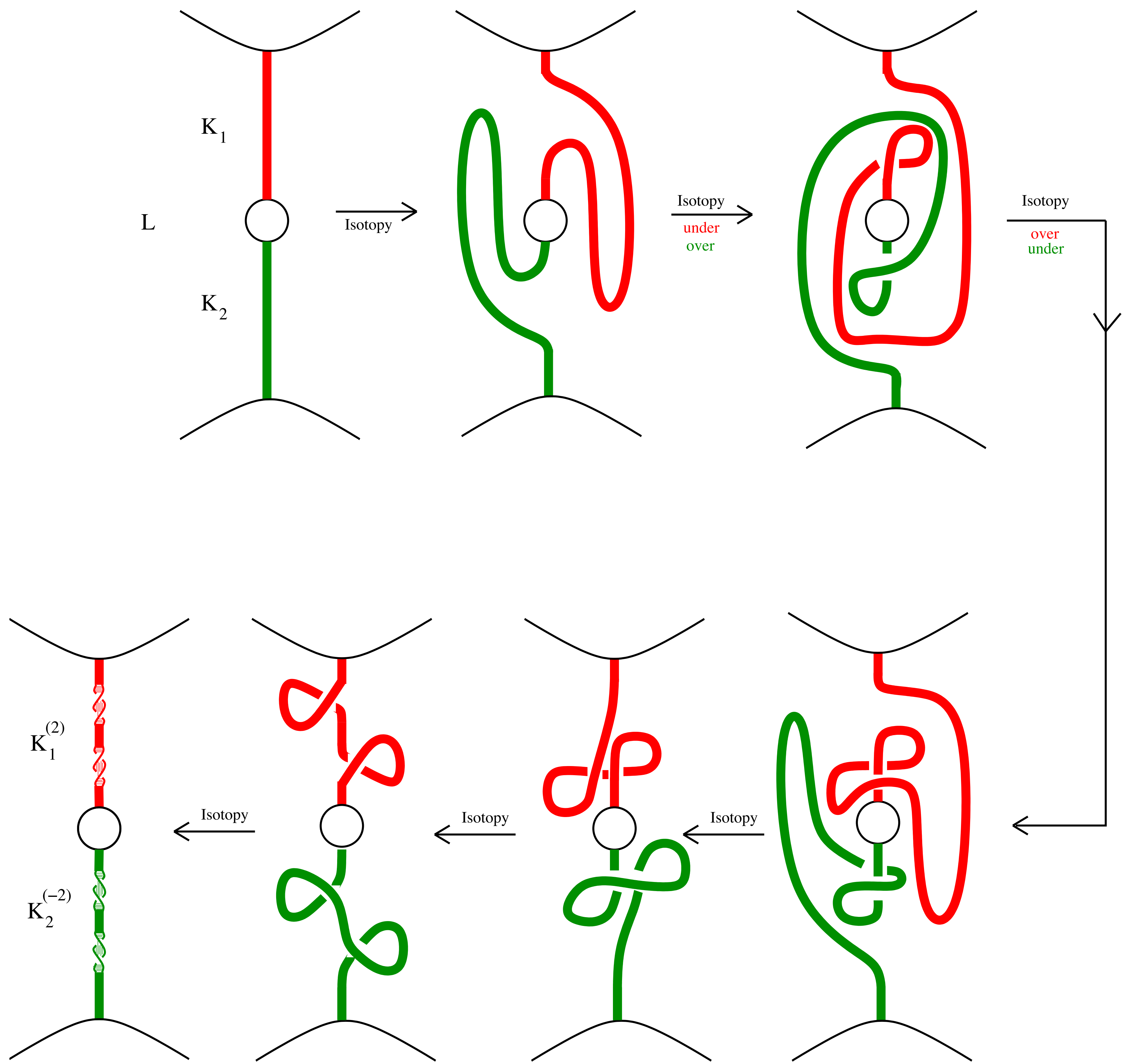
The framing skein module detects the presence of non-separating 2-spheres in M^3.

Theorem (B. - Ibarra - Montoya-Vega - Przytycki - Weeks, 2019)

If M^3 contains a non-separating 2-sphere which a framed link intersects either once or twice (once each by a different component), then an ambient isotopy of M^3 can change the framing of that link only by an even number.



Dirac trick for a knot illustrated using a light bulb



Light bulb trick for a link

Theorem (B. - Ibarra - Montoya-Vega - Przytycki - Weeks, 2019)

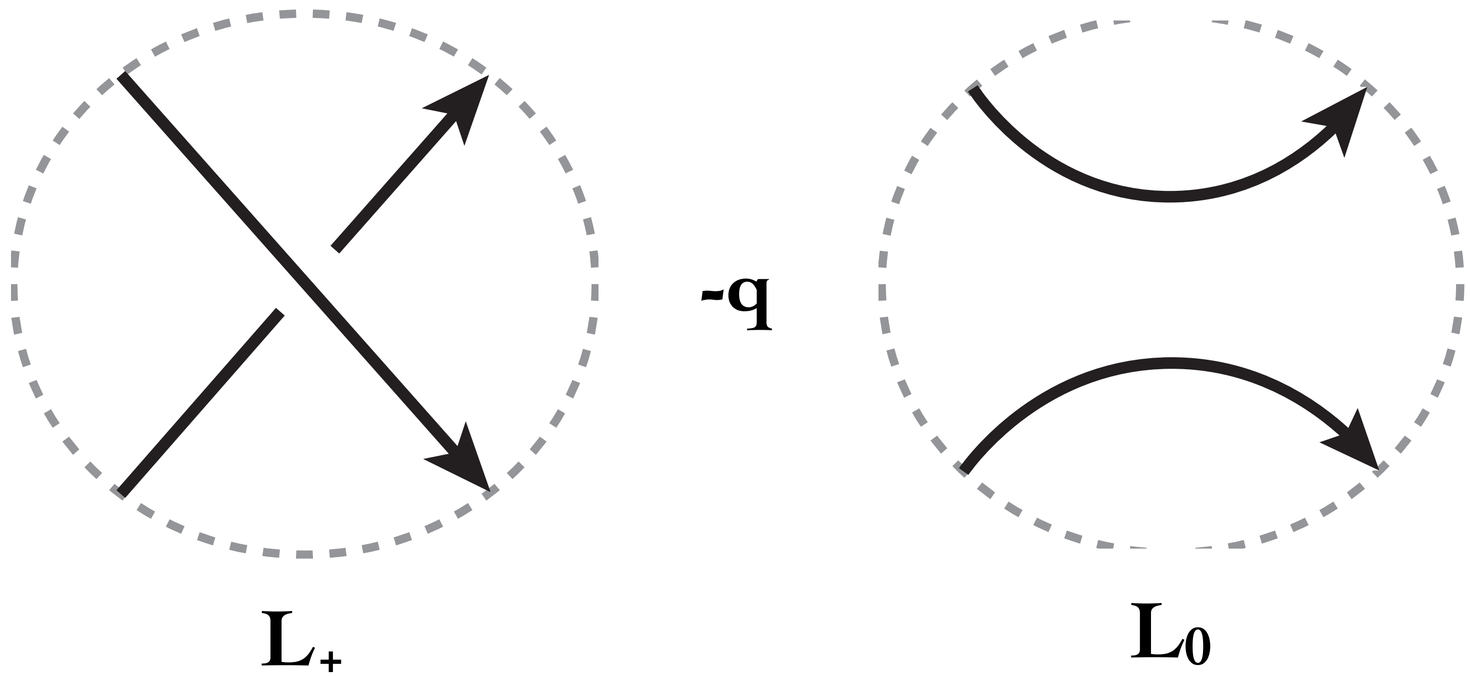
$$\mathcal{S}_0(M^3, q) = \mathbb{Z}[q^{\pm 1}](\mathscr{L}^{fr} \backslash \mathscr{L}^{f}) \oplus \bigoplus_{L \in \mathscr{L}^f} \frac{\mathbb{Z}[q]}{q^2 - 1},$$

where $\mathscr{L}^f$ is the set of ambient isotopy classes of unoriented framed links which intersect any 2-sphere in M^3 transversely at exactly one point.

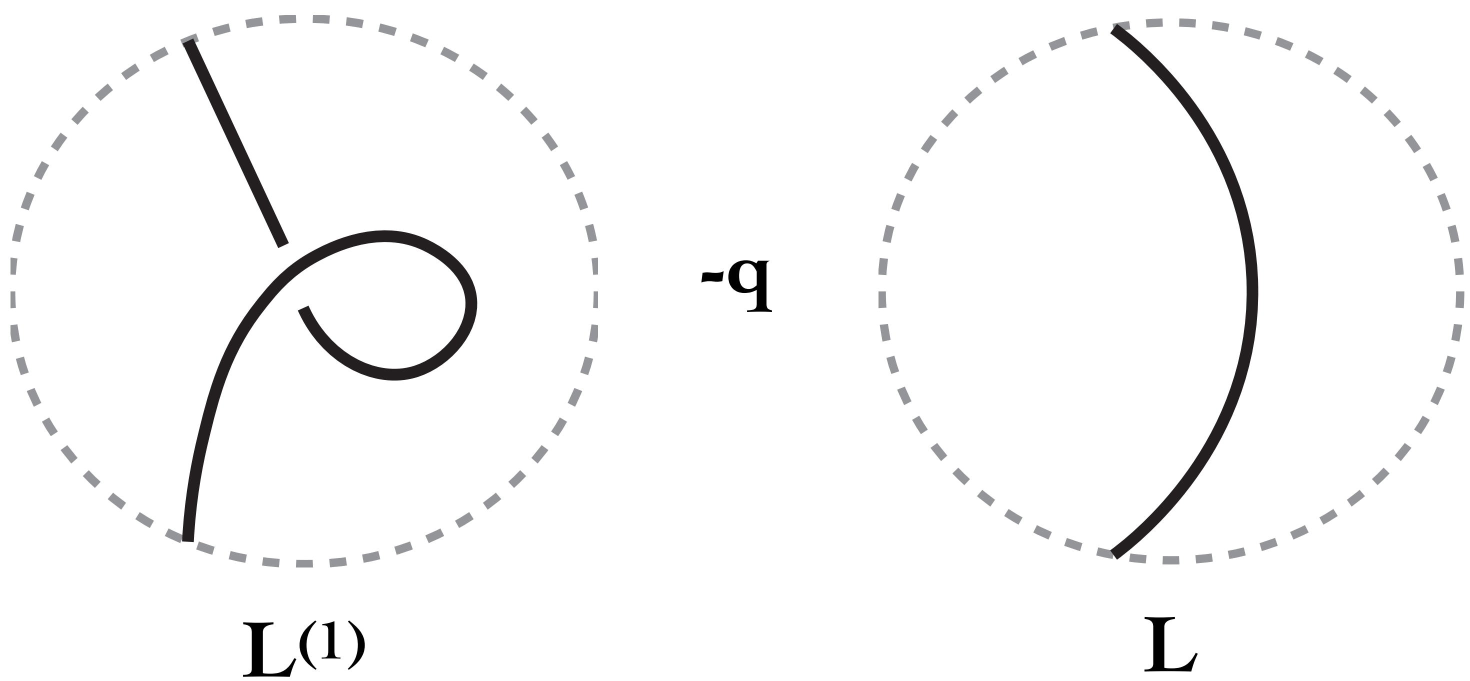
The q-homology Skein Module

Let $R = \mathbb{Z}[q^{\pm 1}]$ and $S_2^{\vec{fr}}$ be the submodule of $R\mathscr{L}^{\vec{fr}}$ generated by the following skein expressions:

Skein Relation:



Framing Relation:



$$\mathcal{S}_2^{\overrightarrow{fr}}(M^3, q) = R\mathcal{L}^{\overrightarrow{fr}} / S_2^{\overrightarrow{fr}}$$

$$\mathcal{S}_2^{\vec{fr}}(M^3, q) = R\mathcal{L}^{\vec{fr}}/S_2^{\vec{fr}}$$

Theorem (Przytycki, 1998)

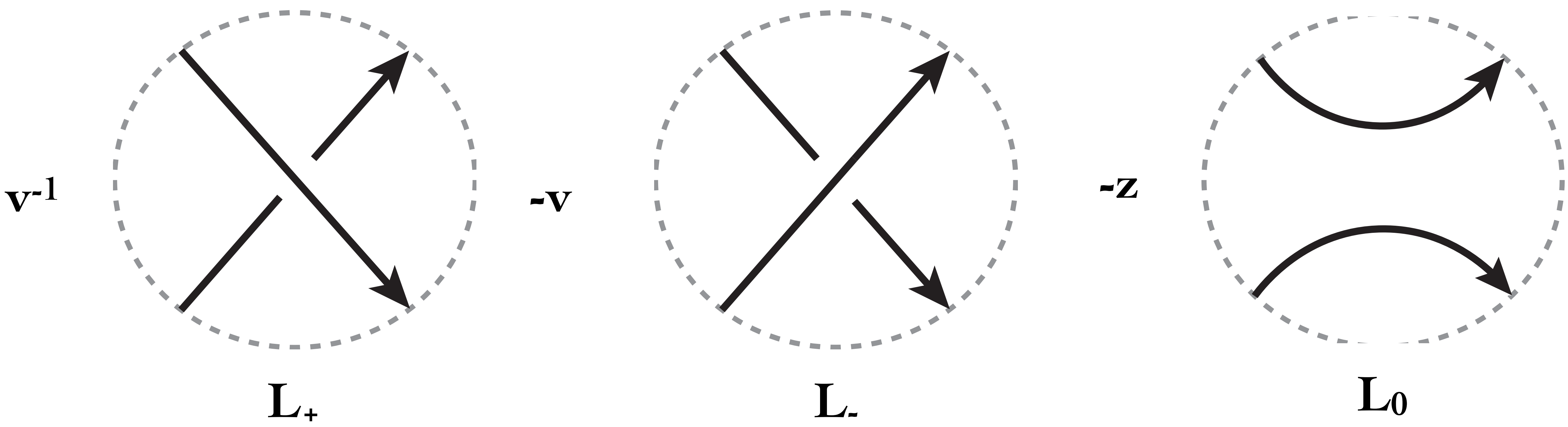
- $\mathcal{S}_2^{\vec{fr}}(M^3, q)$ is a free $\mathbb{Z}[q^{\pm 1}]$-module if M^3 is a rational homology sphere or its compact submanifold.

- The presence of non-separating closed surfaces in M^3 yields torsion in $\mathcal{S}_2^{\vec{fr}}(M^3, q)$.

The Third Skein Module

The Third Skein Module

Let $R = \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ and $\overrightarrow{S}_3$ be the submodule of $R\overrightarrow{\mathscr{L}}$ generated by the following skein expression:



Skein Relation

$$\overrightarrow{\mathcal{S}_3}(M^3) = \overrightarrow{\mathcal{S}_3}(M^3; \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]; v^{-1}L_+ - vL_- - zL_0) = R\overrightarrow{\mathscr{L}}/\overrightarrow{S}_3$$

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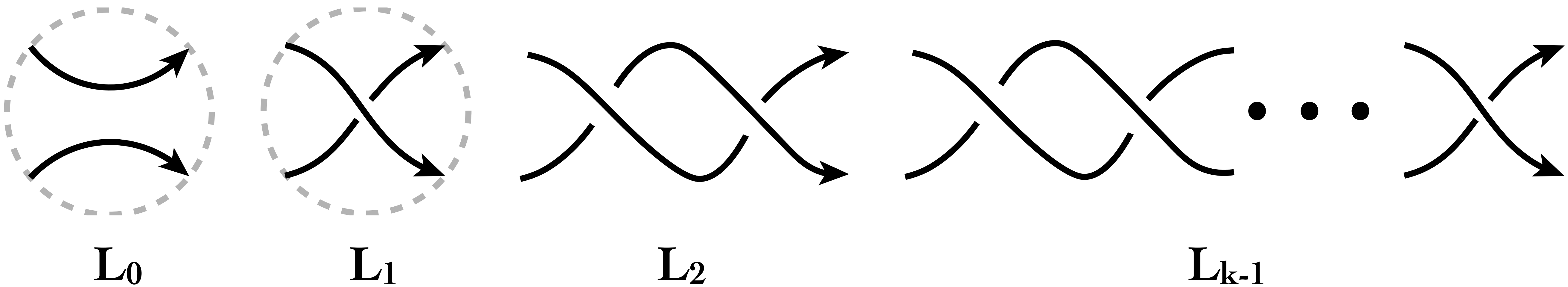
- This skein module is also known as the HOMFLYPT skein module.
- The third skein module of a solid torus was computed by Hoste and Kidwell in 1987 and independently in 1988 by Turaev.
- $\overrightarrow{\mathcal{S}}_3(S^3) = \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ and $\emptyset$ is a generator of the module.
- (Przytycki, 1992): $\overrightarrow{\mathcal{S}}_3(F \times I)$ has an algebra structure and it is isomorphic to the symmetric tensor algebra over conjugacy classes of nontrivial elements of $\pi_1(F)$.
- (Przytycki, 1992): $\overrightarrow{\mathcal{S}}_3(F \times I)$ is an involutory Hopf algebra. Turaev conjectured this in 1991.
- (Sikora, 2001): $\overrightarrow{\mathcal{S}}_3(M^3)$ is related to the algebraic set of $SL(n, \mathbb{C})$ representations of the fundamental group of M^3.

The k^{th} Skein Module

The k[th] Skein Module

Let $\overrightarrow{S}$ be the submodule of $R\overrightarrow{\mathscr{L}}$ generated by the skein expression of the form $r_0 L_0 + r_1 L_1 + \ldots r_{k-1} L_{k-1}$, for $r_0, r_1, \ldots r_{k-1} \in R$.



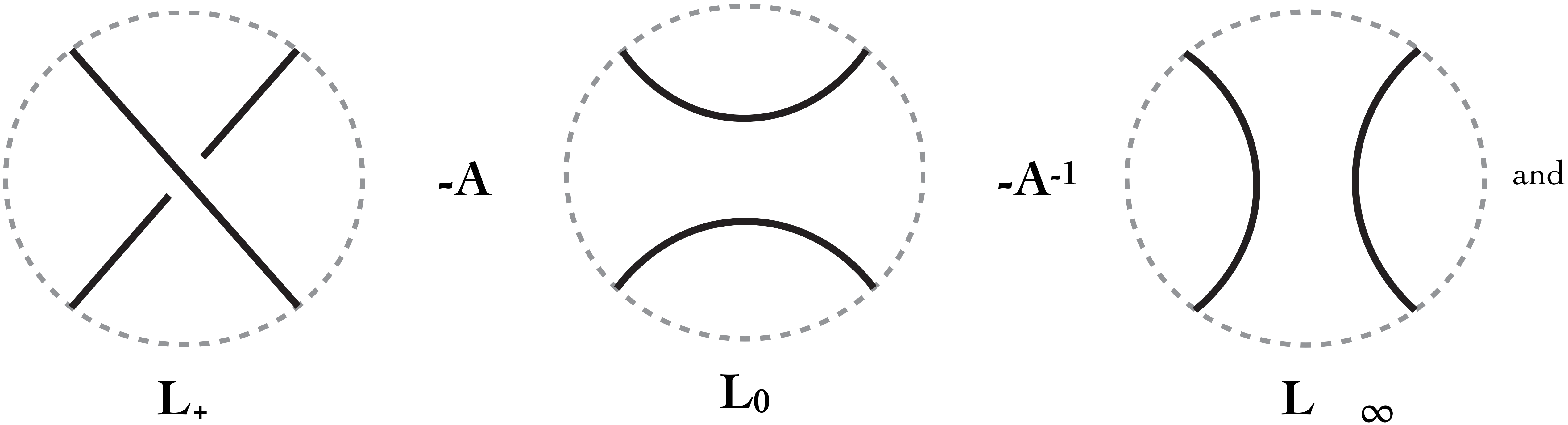
$$\overrightarrow{\mathcal{S}}_k(M^3; R)(r_0, r_1, \ldots, r_{k-1}) = R\overrightarrow{\mathscr{L}} / \overrightarrow{S}$$

OTHER SKEIN MODULES

- The Second Skein module
- The homotopy skein module
- The q-homotopy skein module
- The Fourth, Unoriented Skein Module
- The Vassiliev-Gusarov Skein Module
- The Kauffman skein module
- The Dubrovnik skein module
- The Bar-Natan skein module
- There is also an approach to 4-dimensional skein modules using approaches by Kamada and Kawauchi.

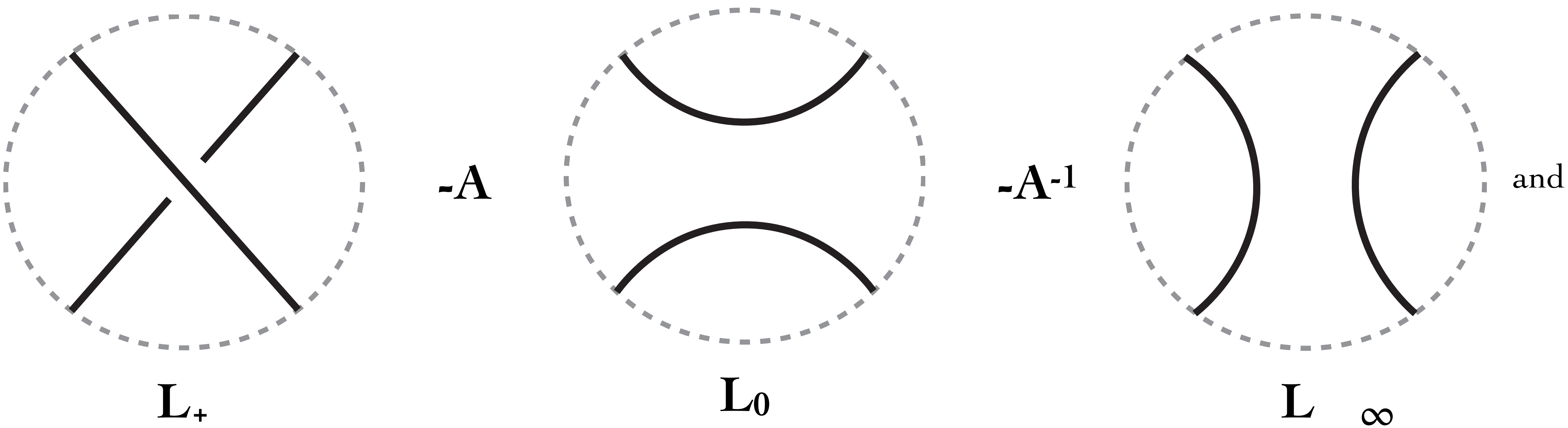
The Kauffman Bracket Skein Module

Let A be an invertible element in R, and $S_{2,\infty}^{sub}$ be the submodule of $R\mathscr{L}^{fr}$ generated by:



$$L \sqcup \bigcirc + (A^2 + A^{-2})L$$

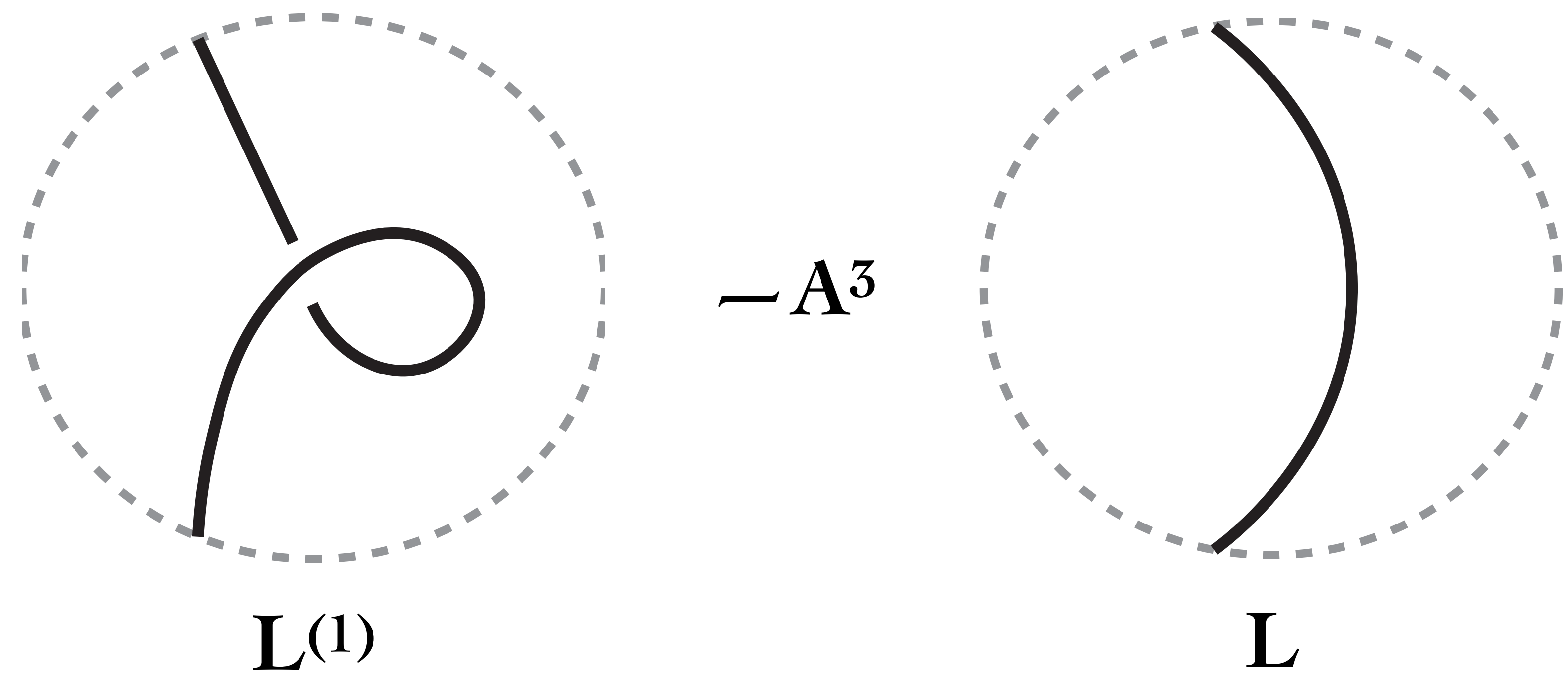
Let A be an invertible element in R, and $S_{2,\infty}^{sub}$ be the submodule of $R\mathcal{L}^{fr}$ generated by:



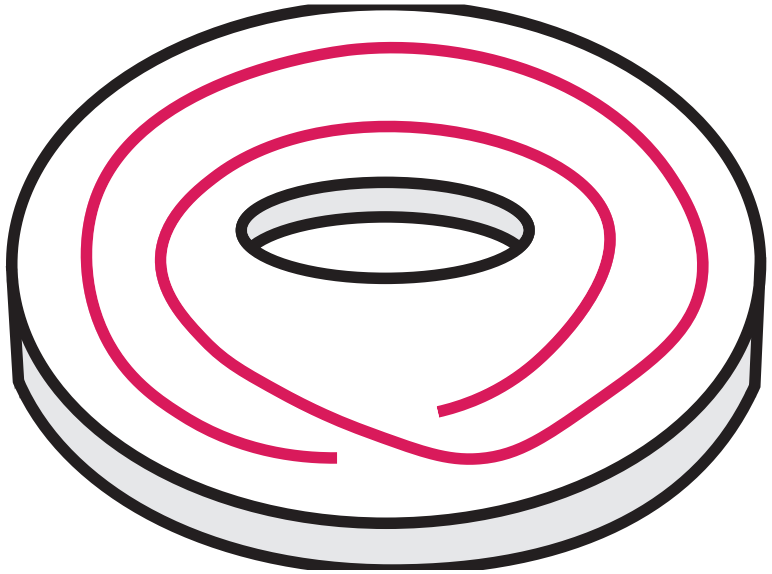
$$L \sqcup \bigcirc + (A^2 + A^{-2})L$$

$$\mathcal{S}_{2,\infty}(M^3; R, A) = R\mathcal{L}^{fr} / S_{2,\infty}^{sub}$$

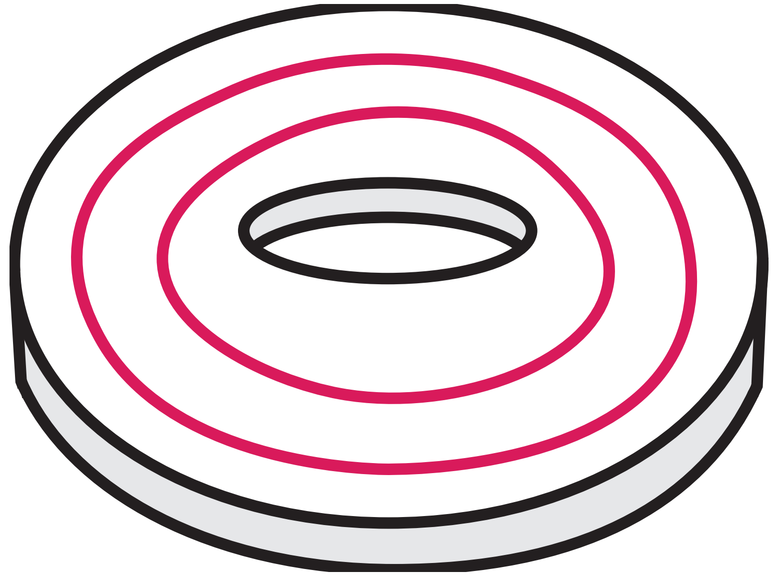
- In $\mathcal{S}_{2,\infty}(M^3; R, A)$



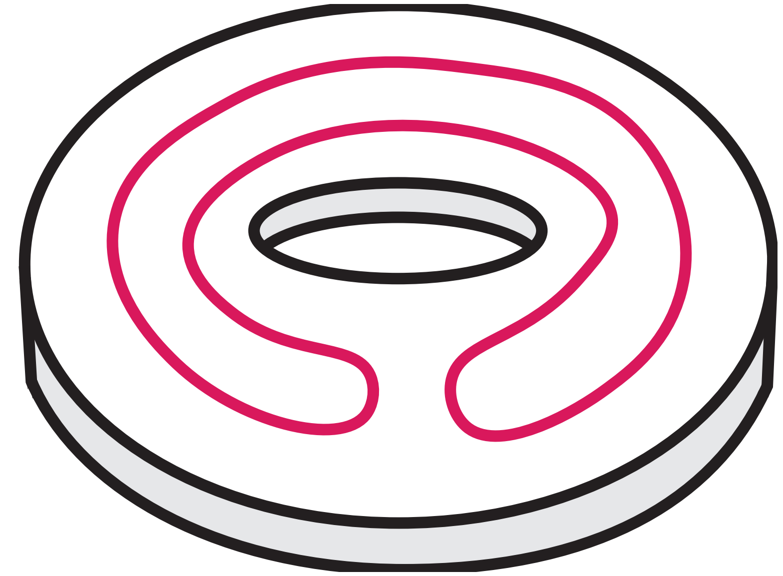
- For simplicity we use the notation when $\mathcal{S}_{2,\infty}(M^3)$ when $R = \mathbb{Z}[A^{\pm 1}]$.



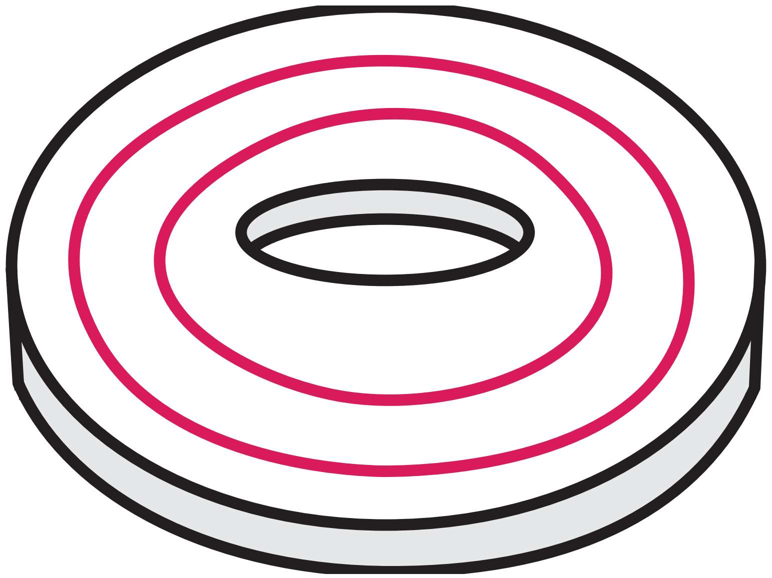
$= A$



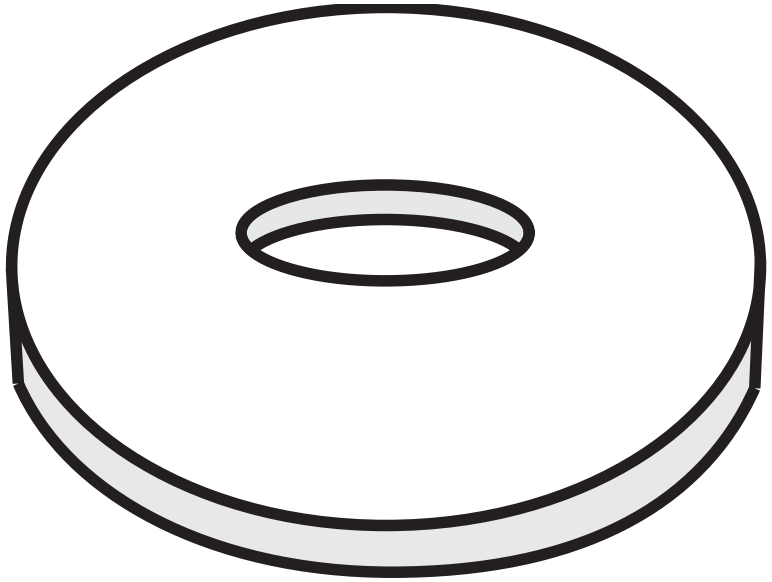
$+ A^{-1}$



$= A$



$+ A^{-1}(-A^{2} - A^{-2})$



$$= Ax^2 + (-A - A^{-3}) \cdot \emptyset$$

Properties of Kauffman Bracket Skein Modules

Przytycki, 1987

- An orientation preserving embedding of 3-manifolds yields a homomorphism of their KBSMs.

- If N^3 is obtained from M^3 by adding a 3-handle to M^3, then the associated embedding yields an isomorphism of the KBSMs of M^3 and N^3.

- Let $(M^3, \partial M^3)$ be a 3-manifold and let γ be a simple closed curve on ∂M^3. Let N^3 be the 3-manifold obtained from M^3 by adding a 2-handle along γ. Then the associated embedding yields an epimorpshim of the KBSMs of M^3 and N^3. Furthermore, the kernel of epimorphism is generated by the relations yielded by 2-handle sliding.

- $\mathcal{S}_{2,\infty}(M_1 \sqcup M_2; R, A) = \mathcal{S}_{2,\infty}(M_1; R, A) \otimes \mathcal{S}_{2,\infty}(M_2; R, A)$

- Let R and R' be commutative rings with unity and $r : R \longrightarrow R'$ be a homomorphism. Then the identity map on L^{fr} induces the following isomorphism of R' (and R) modules:

$$\bar{r} : \mathcal{S}_{2,\infty}(M^3; R, A) \otimes_R R' \longrightarrow \mathcal{S}_{2,\infty}(M^3; R', r(A)) .$$

Theorem(Przytycki, 2000): If M^3, N^3 are compact and $R = \mathbb{Q}(A)$, then

$$\mathcal{S}_{2,\infty}(M^3 \# N^3; R, A) = \mathcal{S}_{2,\infty}(M^3; R, A) \otimes \mathcal{S}_{2,\infty}(N^3; R, A) .$$

Examples of Kauffman Bracket Skein Modules

- $\mathcal{S}_{2,\infty}(S^3) = \mathbb{Z}[A^{\pm 1}]\emptyset$.

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- (Przytycki, 1987): $\mathcal{S}_{2,\infty}(F \times [0,1])$ is a free module generated by the empty link $\emptyset$ and links in F which have no trivial components.

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- (Hoste - Przytycki, 1989): For $p \geq 1$, $\mathcal{S}_{2,\infty}(L(p,q))$ is a free $\mathbb{Z}[A^{\pm 1}]$ -module and it has $\lfloor p/2 \rfloor + 1$ free generators.

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- (Hoste - Przytycki, 1990): $\mathcal{S}_{2,\infty}(S^1 \times S^2)$ is an infinitely generated module.

$$\mathcal{S}_{2,\infty}(S^1 \times S^2)) = \mathbb{Z}[A^{\pm 1}] \oplus \bigoplus_{i=1}^{\infty} \frac{\mathbb{Z}[A^{\pm 1}]}{1 - A^{2i+4}} .$$

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- (Gilmer, 2007): The KBSM of the quaternionic manifold is a finitely generated R-module, where R is the ring $\mathbb{Z}[A^{\pm 1}]$ localized by inverting all the cyclotomic polynomials. The basis consists of five elements.

- (Dabkowski-Mroczkowski, 2009): $\mathcal{S}_{2,\infty}(F_{0,3} \times S^1)$ is an infinitely generated free $\mathbb{Z}[A^{\pm 1}]$ -module.

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- (Carrega, 2016): $\mathcal{S}_{2,\infty}(T^3; \mathbb{Q}(A), A)$ is a finitely generated $\mathbb{Q}(A)$ -module with 9 generators. In 2018, Gilmer showed that these generators are linearly independent.

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- (Detcherry-Wolff, 2019): For any compact closed oriented surface F of genus $g \geq 2$, $\mathcal{S}_{2,\infty}(F \times S^1; \mathbb{Q}(A), A)$ is a finite dimensional $\mathbb{Q}(A)$ -module with dimension $2^{2g+1} + 2g + 1$. Earlier, in 2018, Gilmer and Masbaum had shown that the dimension of this skein module is at least $2^{2g+1} + 2g + 1$.

Witten's Conjecture 2014/2015?

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The Kauffman bracket skein module for any closed oriented 3-manifold over $\mathbb{Q}(A)$, the field of rational functions in the variable A, is always finite dimensional.

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- The first written documentation of this conjecture was in a paper by Carrega (2016).
- Gunningham, Jordan and Safronov resolved this conjecture in August 2019 and proved that it is true.
- The conjecture does not hold when $R = \mathbb{Z}[A^{\pm 1}]$.

Marché's Generalisation of Witten's Conjecture

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Consider closed, compact M^3. Then there exists an integer $d \geq 0$ and finitely generated $\mathbb{Z}[A^{\pm 1}]$ -modules N_k so that

$$\mathcal{S}_{2,\infty}(M^3) = (\mathbb{Z}[A^{\pm 1}])^d \oplus \bigoplus_{k \geq 1} N_k,$$

where N_k is a $(A^k - A^{-k})$ -torsion module for each k .

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- A byproduct of the proof by Detcherry and Wolff is that torsion elements of $\mathcal{S}_{2,\infty}(F \times S^1; \mathbb{Z}[A^{\pm 1}], A)$ are always of $(A^k - A^{-k})$ -torsion for some $k \geq 1$.

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- The conjecture is also true for $\mathcal{S}_{2,\infty}(S^1 \times S^2)$.

- (B., 2020): This conjecture is not true.
- $\mathcal{S}_{2,\infty}(\mathbb{R}P^3 \# \mathbb{R}P^3)$ is a counterexample.
- (Mroczkowski, 2011): Unlike $\mathcal{S}_{2,\infty}(S^1 \times S^2)$, this skein module does not split as a sum of cyclic modules.

The KBSM of the Connected Sums of Handlebodies

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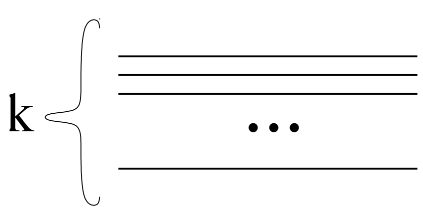
Theorem (Przytycki, 2000)

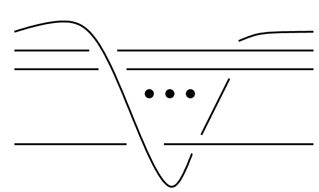
Let $F_{0,n+1}$ be a disc with n holes so that $H_n = F_{0,n+1} \times I$. Then,

$$\mathcal{S}_{2,\infty}(H_n \# H_m) = \mathcal{S}_{2,\infty}(H_{n+m})/\mathcal{I},$$

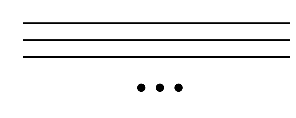
where $\mathcal{I}$ is the ideal generated by expressions $z_k - A^6 u(z_k)$, for any even $k \geq 2$, and z_k are links without contractible components and with geometric intersection number k with a disc D separating H_n and H_m .



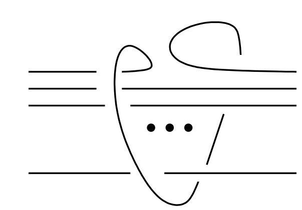
z_k



$u(z_k)$

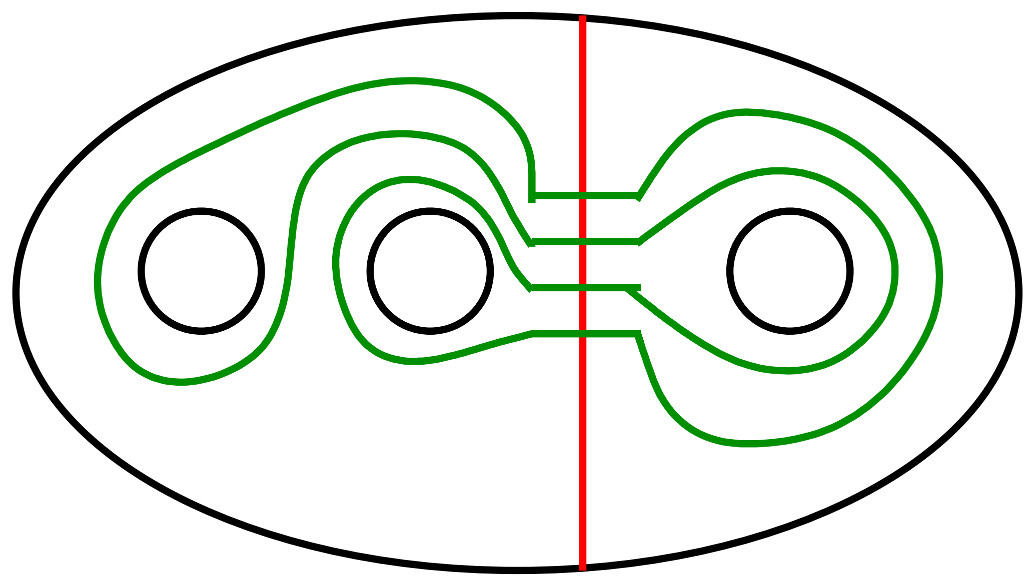


z_k



$sl_{\delta D}(z_k)$

(B. - Przytycki, 2020): This theorem is incorrect. The counterexample is given by $H_n \# H_m,\ n \geq 2, m \geq 1$ and we showed that the ideal $\mathcal{I}$ should be replaced by a strictly bigger ideal to obtain the equality stated in the theorem.



Curve system in $H_2 \# H_1$ leading to the counterexample

- The case for $H_1 \# H_1$ is work in progress.

The Kauffman Bracket Skein Algebra

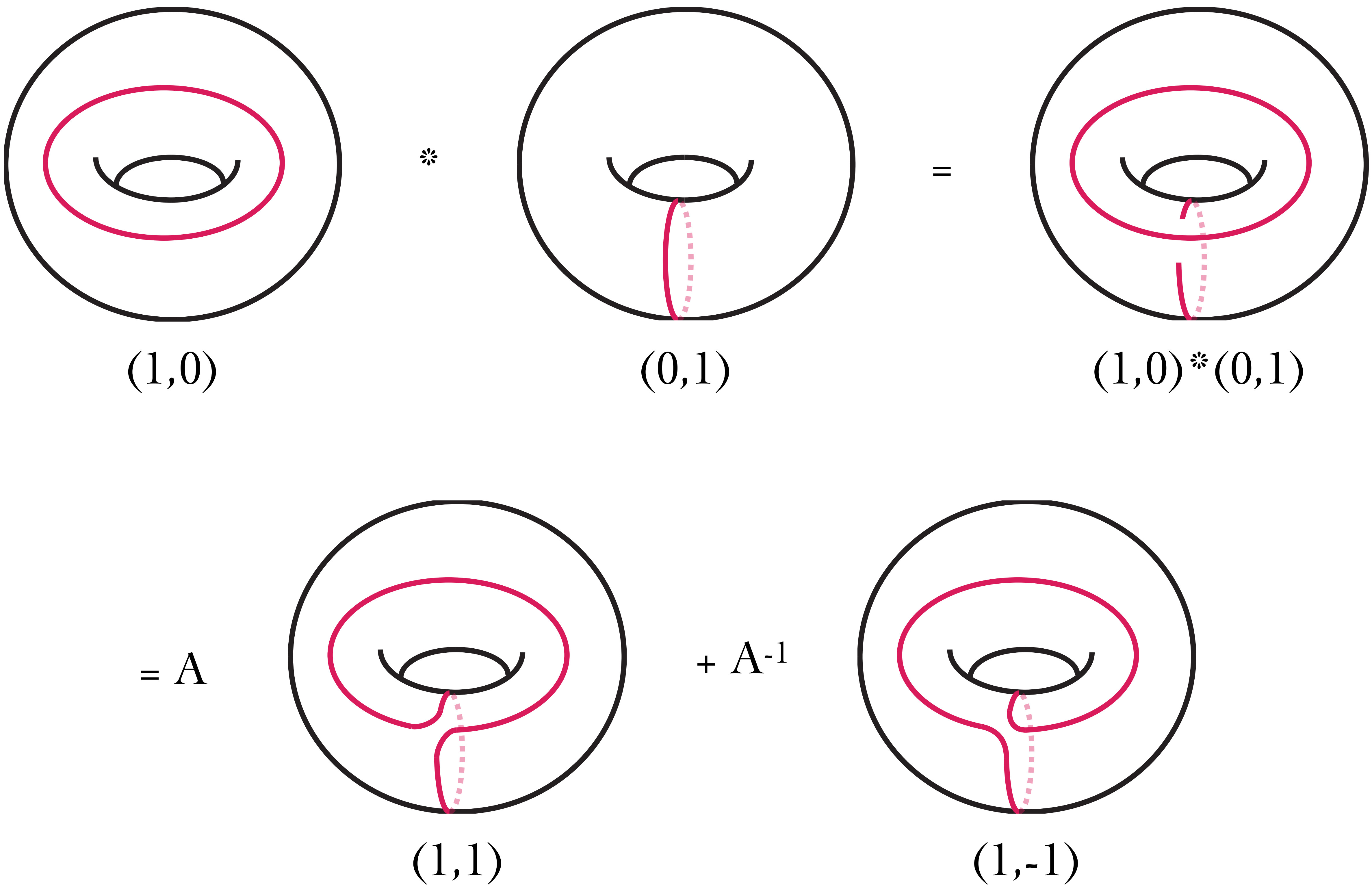
- The KBSM of surface I-bundles can be enriched with an algebra structure.

- The empty link serves as the multiplicative identity.

- $L_1 \cdot L_2 :=$ place L_1 over L_2 that is, $L_1 \subset F \times (\frac{1}{2}, 1)$ and $L_2 \subset F \times (0, \frac{1}{2})$

- We denote this algebra as $\mathcal{S}^A(F \times I; R, A)$. For brevity, we use the notation $\mathcal{S}^A(F \times I)$ when $R = \mathbb{Z}[A^{\pm 1}]$.

- This can be generalised to some unoriented surfaces and we can work with the twisted I-bundles over those surfaces.



(1,0)
*
(0,1)
=
(1,0)*(0,1)
= A
(1,1)
+ A-1
(1,-1)

A Few Remarks

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- The embedding $i : F \hookrightarrow F'$ of oriented surfaces induces a homomorphism of skein algebras $i_* : \mathcal{S}^A(F \times I; R, A) \longrightarrow \mathcal{S}^A(F' \times I; R, A)$.

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- The embedding $i : F \hookrightarrow F'$ of oriented surfaces induces a homomorphism of skein algebras $i_* : \mathcal{S}^A(F \times I; R, A) \longrightarrow \mathcal{S}^A(F' \times I; R, A)$.

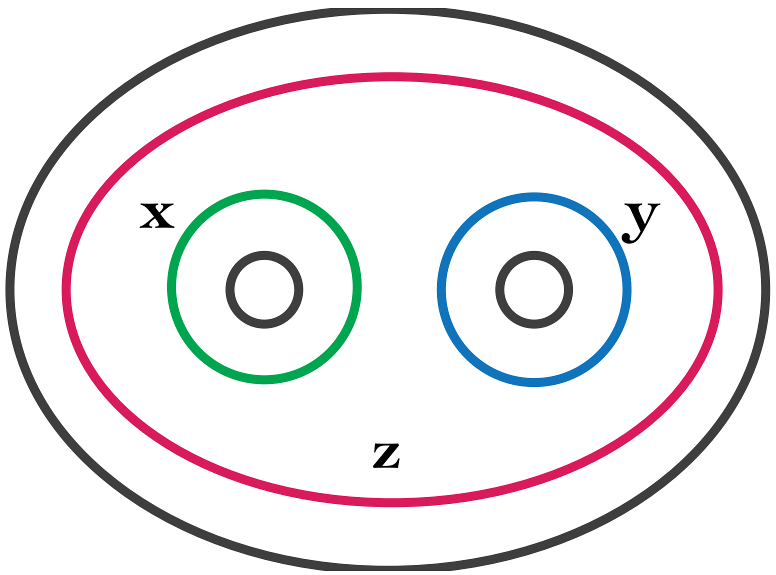
- It is possible that the I-bundles over two non-homeomorphic surfaces F_1 and F_2 are homeomorphic, e.g. the once punctured torus and the pair of pants.

  In this case, $\mathcal{S}_{2,\infty}(F_1 \times I; R, A) \cong \mathcal{S}_{2,\infty}(F_2 \times I; R, A)$, but $\mathcal{S}^A(F_1 \times I; R, A) \ncong \mathcal{S}^A(F_2 \times I; R, A)$.

Examples of Kauffman Bracket Skein Algebras

Theorem (Bullock - Przytycki, 2000)

- $\mathcal{S}^A(S^2 \times I) \cong \mathcal{S}^A(F_{0,1} \times I) \cong \mathbb{Z}[A^{\pm 1}]$.
- $\mathcal{S}^A(F_{0,2} \times I) \cong \mathbb{Z}[A^{\pm 1}][x]$.
- $\mathcal{S}^A(F_{0,3} \times I) \cong \mathbb{Z}[A^{\pm 1}][x, y, z]$.



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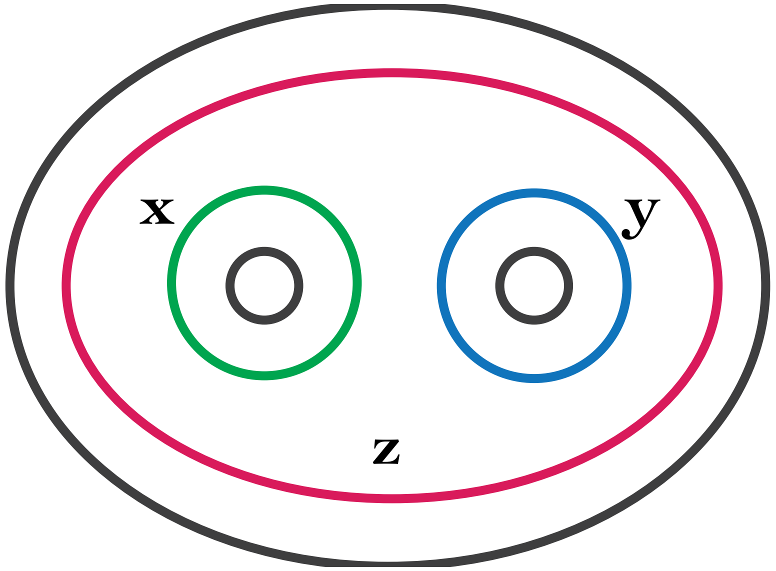

- Chebyshev polynomials of the 2nd kind $S_i(x)$ form a basis for $\mathcal{S}^A(F_{0,2} \times I)$. *Thread* links in the annulus with the polynomials, that is, replace each component of the link with $S_i(x)$.

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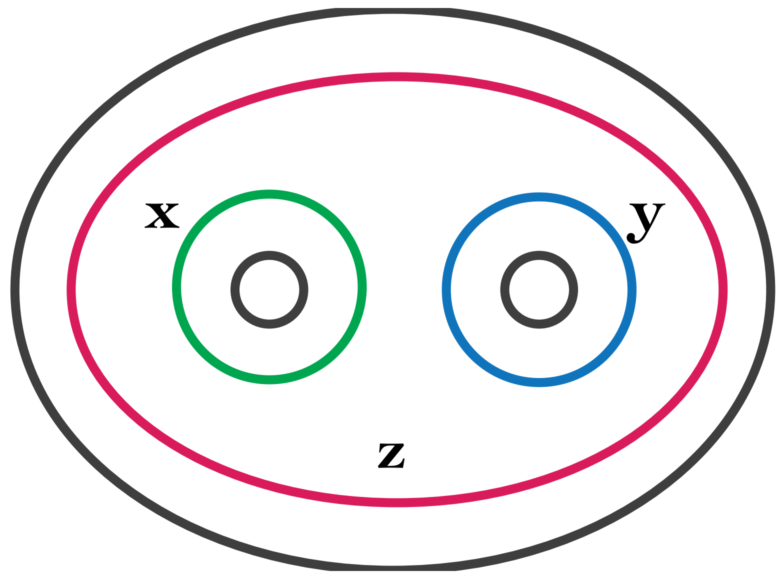

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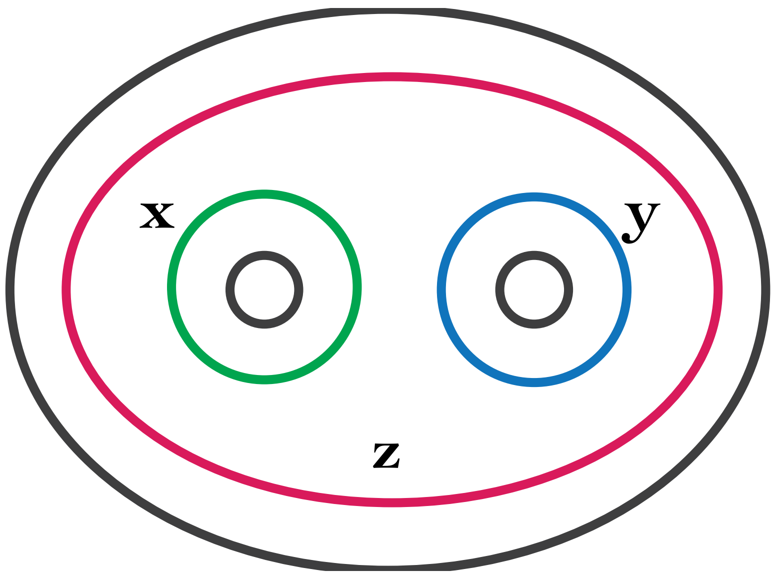

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Theorem (Bullock - Przytycki, 2000)

- $\mathcal{S}^A(F_{1,1} \times I) \cong \dfrac{\mathbb{Z}[A^{\pm 1}]\{(1,0),(0,1),(1,1)\}}{equations\ 1,2,\ and\ 3}$.
- $\mathcal{S}^A(F_{1,0} \times I) \cong \dfrac{\mathbb{Z}[A^{\pm 1}]\{(1,0),(0,1),(1,1)\}}{equations\ 1,2,3,\ and\ 4}$.

Equation 1: $A[(1,0)*(0,1)] - A^{-1}[(0,1)*(1,0)] = (A^2 - A^{-2})(1,1)$

Equation 2: $A[(0,1)*(1,1)] - A^{-1}[(1,1)*(0,1)] = (A^2 - A^{-2})(1,0)$

Equation 3: $A[(1,1)*(1,0)] - A^{-1}[(1,0)*(1,1)] = (A^2 - A^{-2})(0,1)$

Equation 4: $A^2[(1,1)^2 + (1,0)^2] + A^{-2}(0,1)^2 - 2(A^2 + A^{-2}) - A((1,0)*(0,1)*(1,1)) = 0$

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$$(p,q)_T = T_d\left(\frac{p}{d}, \frac{q}{d}\right)$$

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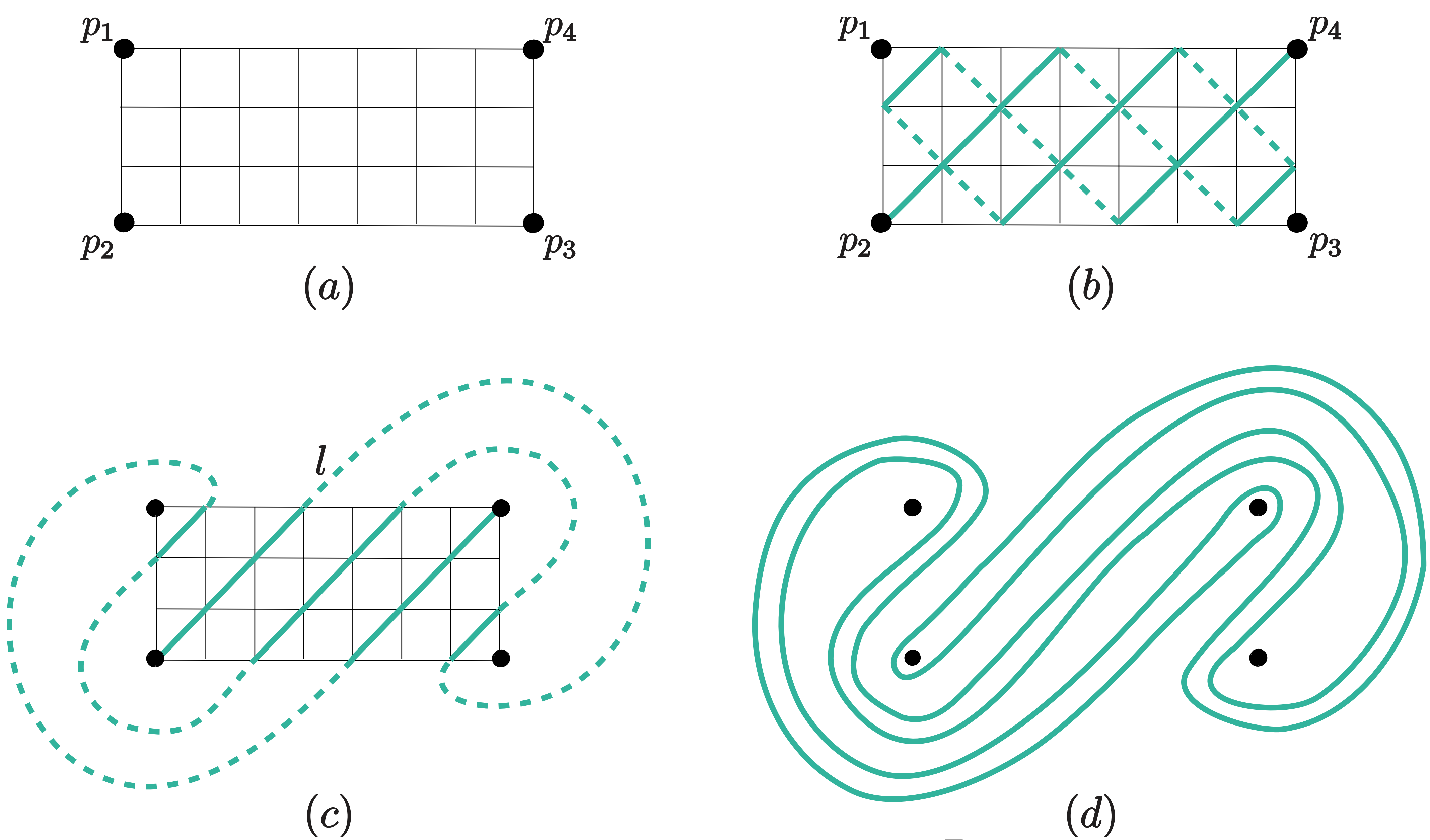
Theorem (Frohman - Gelca, 2000)

$$(p, q)_T * (r, s)_T = t^{ps-qr}(p+q, r+s)_T + t^{-(ps-qr)}(p-q, r-s)_T .$$

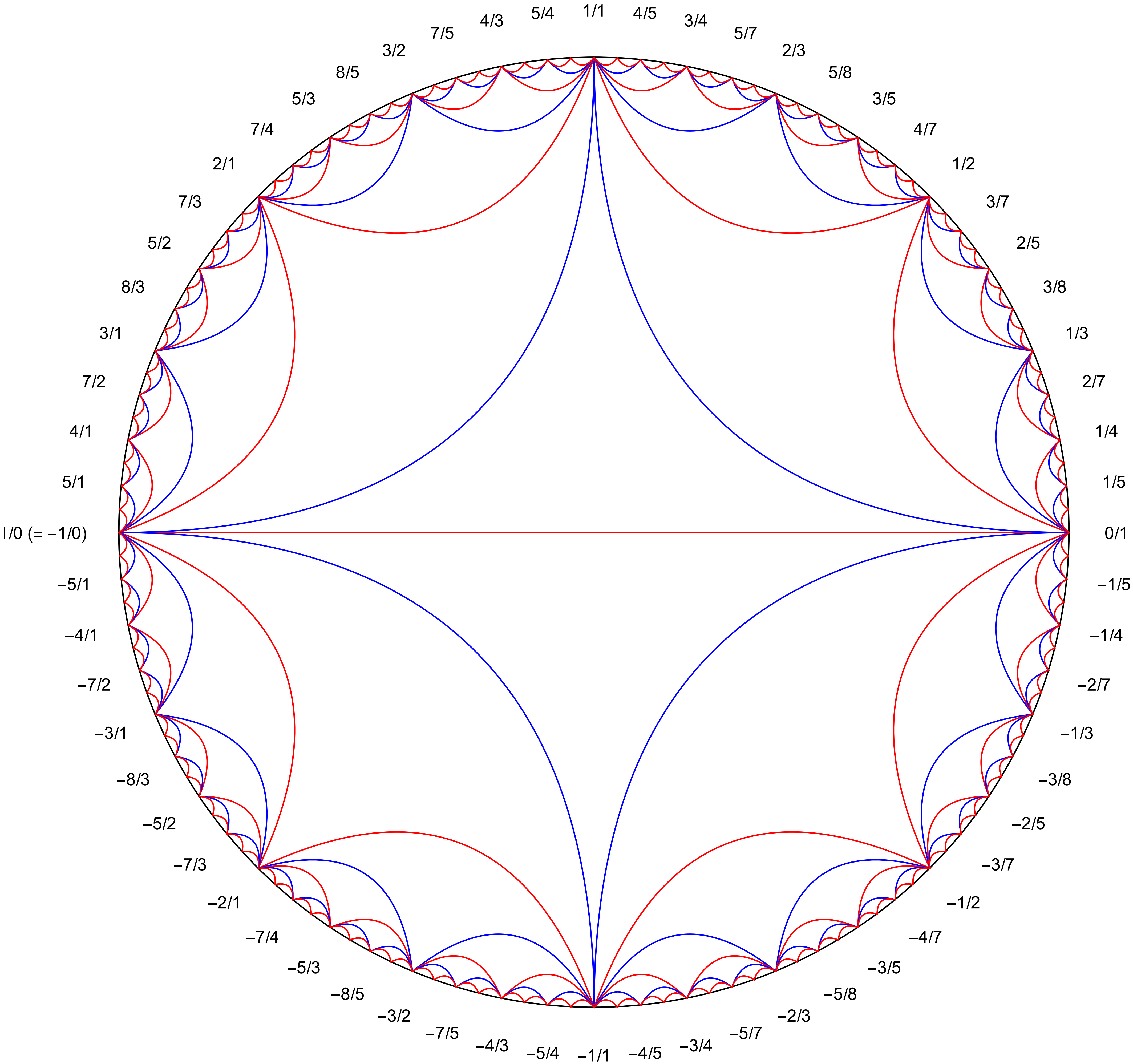
The KBSA of the Four-Punctured Sphere



The curve (3,7) = $\frac{7}{3}$

- (Bullock - Przytycki, 2000): $S^A(F_{0,4} \times I)$ is generated by a_1, a_2, a_3, a_4, (1,0), (0,1), and (1,1).

- In 2018 B., Mukherjee, Przytycki, Silvero and Wang provided an algorithm to compute the product of any two elements in the skein algebra, and give explicit formulas for some infinite families of curves. The algorithm is based on the action of the generators of the mapping class group of $F_{0,4}$ on the basis elements of $S^A(F_{0,4} \times I)$ and uses an adapted version of the Euclidean algorithm.



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4/3
5/4

The Positivity Conjecture

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- A basis $\mathscr{B}$ for $\mathcal{S}^A(F \times I)$ over $\mathbb{Z}[A^{\pm 1}]$ is positive if the structure constants for multiplication lie in $\mathbb{Z}_{\geq 0}[A^{\pm 1}]$, that is, for any x_i and x_j in $\mathscr{B}$ their product $x_i \cdot x_j = \sum_k a_k x_k$, where $a_k \in \mathbb{Z}_{\geq 0}[A^{\pm 1}]$ and $x_k \in \mathscr{B}$.

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- Which bases of the KBSA of surface I-bundles are positive?

- This question was first posed by Fock and Goncharov in 2006 while working on cluster algebras.

Theorem (Thurston, 2015)

Consider the Chebyshev polynomial $\hat{T}_n$ normalised so that the initial conditions are $\hat{T}_0(x) = 1,\ \hat{T}_1(x) = x,\ \hat{T}_2(x) = x^2 - 2$. If $R = \mathbb{Z}$ and $A = 1$, then $(\hat{T}_n)$ is positive on any surface.

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Theorem (Lé - Thurston - Yu, 2019)

Any positive basis (P_n) is bounded below by $(\hat{T}_n)$ and above by (S_n) if the surface has at least 4 punctures or genus at least 1.

Example of the KBSA of a Non-Oriented Surface

- $\mathcal{S}_{2,\infty}(L(2,1))$ has an algebra structure and,

$$\mathcal{S}^{A}(L(2,1)) \cong \mathbb{Z}[A^{\pm 1}][\alpha]/\left(\alpha^2 - A^3\frac{A^4 - A^{-4}}{A - A^{-1}}\right)$$

where α denotes an orientation reversing curve in $\mathbb{R}P^2$.

Properties of Kauffman Bracket Skein Algebras

Theorem (Bullock 1999, Przytycki - Sikora, 2000)

If F is a compact oriented surface, then $\mathcal{S}^A(F \times I)$ is finitely generated, and the minimal number of generators is $2^{rank(H_1(F))} - 1$.

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Theorem (Przytycki - Sikora, 2019)

$\mathcal{S}^A(F \times I; R, A)$ has no zero divisors, provided R has no zero divisors. This in turn implies that if R has no nilpotent elements then neither does $\mathcal{S}^A(F \times I; R, A)$.

Theorem (Przytycki - Sikora, 2019)

Let F be an unoriented surface with even negative Euler characteristic, then $\mathcal{S}^A(F \hat{\times} I; R, \pm 1)$ has no zero divisors, provided R has no zero divisors. This result does not hold when F is the Klein bottle.

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If $A^{4n} - 1$ is not a zero divisor in R for any n > 0, then the centre of the KBSA $\mathcal{S}^A(F \times I; R, A)$ is a subalgebra generated by the boundary components of F.

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Theorem (Frohman - Kania-Bartoszynska - Lé, 2019)

Let F be a finite type surface, then $\mathcal{S}^A(F \times I; \mathbb{C}, A)$ is finitely generated as a module over its centre.

Connection to the $SL(2,\mathbb{C})$ Variety

- An $SL(2,\mathbb{C})$ representation of the fundamental group of M^3 is a homomorphism $\rho : \pi_1(M^3) \longrightarrow SL(2,\mathbb{C})$. The character of a representation is the composition $\chi_\rho = trace \circ \rho$.

- A closed algebraic set X in $\mathbb{C}^m$ is the common zero set of an ideal of polynomials in $\mathbb{C}[\gamma_1, \ldots, \gamma_m]$. The elements of $\mathbb{C}[\gamma_1, \ldots, \gamma_m]$ are polynomial functions on X, and the functions γ_i are coordinates on X. The quotient of $\mathbb{C}[\gamma_1, \ldots, \gamma_m]$ by the ideal of polynomials vanishing on X is called the coordinate ring of X.

- Denote the set of all characters by $X(M^3)$. For each $\gamma \in \pi_1(M^3)$ consider the function $t_\gamma : X(M^3) \longrightarrow \mathbb{C}$ given by $\chi_\rho \longmapsto \chi_\rho(\gamma)$.

Theorem (Culler - Shalen, 1983)

There exists a finite set of elements $\{\gamma_1, \ldots, \gamma_m\}$ in $\pi_1(M^3)$ such that every t_γ is an element of the polynomial ring $\mathbb{C}[t_{\gamma_1}, \ldots, t_{\gamma_m}]$.

Theorem (Culler - Shalen, 1983)

If every t_γ is an element of $\mathbb{C}[t_{\gamma_1}, \ldots, t_{\gamma_m}]$, then $X(M^3)$ is a closed algebraic subset of $\mathbb{C}^m$.

Denote the coordinate ring of $X(M^3)$ by $\mathscr{R}(M^3)$. Then $\mathscr{R}(M^3)$ lies in $\mathbb{C}^{X(M^3)}$, the algebra of functions from $X(M^3)$ to $\mathbb{C}$.

Theorem (Bullock, 1997)

Let $\widetilde{\phi} : \mathbb{C}\mathscr{L}^{fr} \longrightarrow \mathbb{C}^{X(M)}$ be the linear map which sends each knot to the negative of its naturally induced function and each link to the product of the images of its components. Then $\widetilde{\phi}$ descends to a well defined map of algebras $\phi : \mathscr{S}^A(M^3; \mathbb{C}, -1) \longrightarrow \mathbb{C}^{X(M^3)}$. The image of this map is the coordinate ring $\mathscr{R}(M^3)$ of $X(M^3)$ and its kernel is the ideal consisting of nilpotent elements of $\mathscr{S}^A(M^3; \mathbb{C}, -1)$.

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Theorem (Bullock 1997, Przytycki - Sikora 2019)

$\mathcal{S}^A(F \times I; \mathbb{C}, -1)$ is isomorphic to the coordinate ring of the $SL(2,\mathbb{C})$ character variety of the fundamental group of the surface.