Gram Determinants Motivated By Knot Theory

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Overview

- The Kauffman Bracket Polynomial
- Relative Kauffman Bracket Skein Modules
- The Temperley - Lieb Algebra
- The Gram Determinant of Type A
- The Gram Determinant of Type B
- The Gram Determinant of Generalized Type A
- Jones - Wenzl Idempotents
- Theta Nets
- The Gram Determinant of Type Mb

The Kauffman Bracket Polynomial

The Kauffman bracket polynomial is a function from the set of unoriented link diagrams in $\mathbb{R}^2$ or $\mathbb{S}^2$ to the ring of Laurent polynomials with integer coefficients in an indeterminate A. It maps a diagram D to $\langle\, D \,\rangle \in \mathbb{Z}[A^{\pm 1}]$ and is characterized by the following axioms:

(1) $\langle \bigcirc \rangle = 1$,

(2) $\langle \bigcirc \sqcup D \rangle = d\langle D \rangle = -(A^2 + A^{-2})\langle D \rangle$, and

(3) $\langle \diagdown\!\!\!\!\diagup \rangle = A\langle \asymp \rangle + A^{-1}\langle\,)\,(\,\rangle$.

The last axiom is known as the Kauffman bracket skein relation.

- The Kauffman bracket polynomial is an invariant of framed links.

Relative Kauffman Bracket Skein Modules (RKBSM)

Definition

Let M be an oriented 3 - manifold and $\{x_1, x_2, \ldots, x_{2n}\}$ be a set of $2n$ framed points on ∂M. Let $\mathcal{L}_{fr}(2n)$ be the set of all relative framed links in $(M, \partial M)$ considered up to ambient isotopy keeping ∂M fixed, such that $L \cap \partial M = \partial L = \{x_i\}$.

Let R be a commutative ring with unity, A an invertible element in R, and $S_{2,\infty}(2n)$, the submodule of $R\mathcal{L}_{fr}(2n)$, generated by all the Kauffman bracket skein relations.

Then, the relative Kauffman bracket skein module of M is the quotient:

$$\mathcal{S}_{2,\infty}(M, \{x_i\}_1^{2n}; R, A) = \frac{R\mathcal{L}_{fr}(2n)}{S_{2,\infty}(2n)}.$$

Relative Kauffman Bracket Skein Modules Continued

> **Theorem (Przytycki)**
> *Let F be a surface and $M = F \times I$ (or $F \hat{\times} I$, if F is unoriented), such that $\partial F \neq \emptyset$. Then, $\mathcal{S}_{2,\infty}(M, \{x_i\}_1^{2n}; R, A)$ is a free R-module whose basis consists of relative links in F, without trivial components.*

Corollary

1. $\mathcal{S}_{2,\infty}(D^2 \times I, \{x_i\}_1^{2n}; R, A)$ *is a free R-module with* $c_n = \frac{1}{n+1}\binom{2n}{n}$ *basic elements. Here c_n denotes the n^{th} Catalan number.*
2. $\mathcal{S}_{2,\infty}(A^2 \times I, \{x_i\}_1^{2n}; R, A)$ *is a free $R[z]$ - module with* $\binom{2n}{n}$ *basic elements, where z denotes the homotopically non - trivial curve on the annulus.*

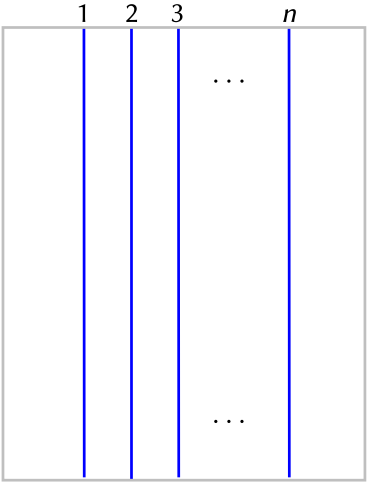
- The relative Kauffman bracket skein module of $D^2 \times I$ can be equipped with an algebra structure which leads to the classical Temperley - Lieb algebra.

Definition

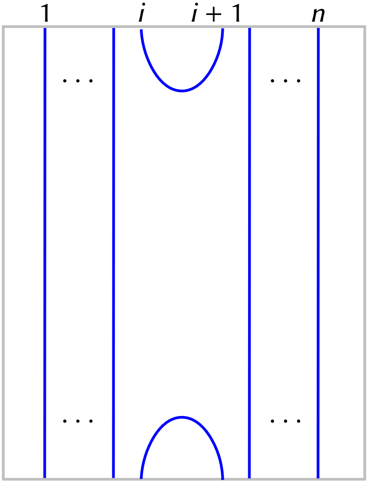
Let R be a commutative ring with unity and an invertible element A. Further, let $d = -A^{-2} - A^2 \in R$. The Temperley - Lieb Algebra, denoted by $TL_n(d)$, is an R - algebra generated by n elements, $\{\mathbb{1}, e_1, e_2, \ldots, e_{n-1}\}$ with relations:

1. $e_i^2 = de_i$ for $1 \leq i \leq n-1$,
2. $e_ie_je_i = e_i \ \forall |i-j| = 1$, and
3. $e_ie_j = e_je_i \ \forall |i-j| > 1$.

The Generators of $TL_n(d)$



(a) The identity element $\mathbb{1}$



(b) e_i, $1 \leq i \leq n-1$

The Gram Determinant of Type A

Consider the disc D^2 with $2n$ framed points on its boundary. Let $\mathcal{A}_n = \{a_1, a_2, \ldots, a_{c_n}\}$ be the set of all diagrams with crossingless connections, up to ambient isotopy, between the $2n$ framed points in D^2. Define a bilinear form $\langle\ ,\ \rangle$ in the following way:

$$\langle\ ,\ \rangle : \mathcal{S}_{2,\infty}(D^2 \times I, \{x_i\}_1^{2n}) \times \mathcal{S}_{2,\infty}(D^2 \times I, \{x_i\}_1^{2n}) \longrightarrow R.$$

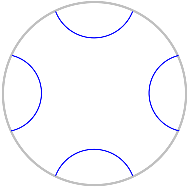
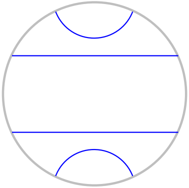
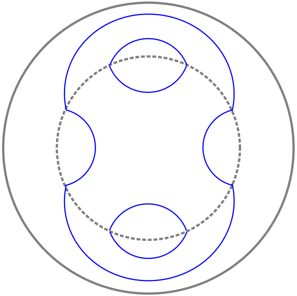
For $a_i, a_j \in \mathcal{A}_n$, glue a_i with the inversion of a_j along the marked circle, respecting the labels of the framed points. The resulting picture is that of a disc with disjoint null homotopic circles. Thus, we define, $\langle a_i, a_j\rangle = d^m$ where m denotes the number of these circles.

Definition

The Gram matrix of type A is defined as $G_n^A = (\langle a_i, a_j\rangle)_{1\leq i,j\leq c_n}$. Its determinant D_n^A is called the Gram determinant of type A.

Type A Bilinear Form when $n = 4$

Example

$\Big\langle$  ,  $\Big\rangle =$  $= d^3$

Theorem (Westbury, Di Francesco, Cai)

Let $R = \mathbb{Z}[A^{\pm 1}]$. *Then,*

$$D_n^A(d) = \prod_{i=1}^{n} \left(\frac{\Delta_i}{\Delta_{i-1}}\right)^{\alpha_i}, \text{ where } \Delta_i = (-1)^i \frac{A^{2i+2} - A^{-2i-2}}{A^2 - A^{-2}}$$

$$\text{and } \alpha_i = \binom{2n}{n-i} - \binom{2n}{n-i-1}.$$

Note that $\Delta_1 = -A^2 - A^{-2} = d$. Furthermore, $\Delta_i(d)$ is the Chebyshev polynomial of the second kind.

- Cai used Jones - Wenzl idempotents and theta nets to construct a new basis of the Temperley - Lieb algebra and provided a proof for the formula of the Gram determinant of Type A.

The Gram Determinant of Type B

Let A^2 be an annulus with $2n$ marked points on its outer boundary. Let $\mathcal{B}_n = \{b_1, b_2, \ldots, b_{\binom{2n}{n}}\}$ be the set of all diagrams of crossingless connections between these $2n$ points. Define a bilinear form $\langle\ ,\ \rangle$ in the following way:

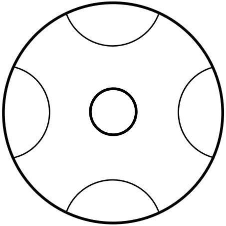
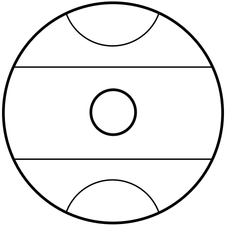
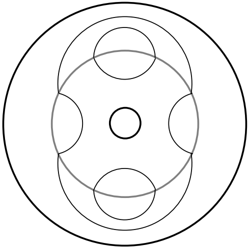
$$\langle\ ,\ \rangle : \mathcal{S}_{2,\infty}(A^2 \times I, \{x_i\}_1^{2n}; R, A) \times \mathcal{S}_{2,\infty}(A^2 \times I, \{x_i\}_1^{2n}; R, A) \longrightarrow R[z].$$

Given $b_i, b_j \in \mathcal{B}_n$, glue b_i with the inversion of b_j along the marked circle, respecting the labels of the marked points. The resulting picture has disjoint circles which are either homotopically non - trivial or null homotopic. Then, $\langle b_i, b_j \rangle = z^k d^m$, where k and m denote the number of these circles, respectively.

Definition

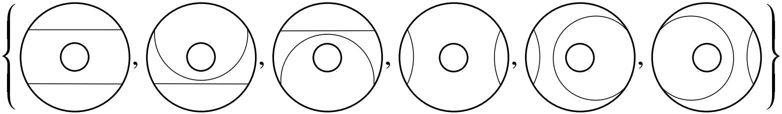
The Gram matrix of type B is defined as $G_n^B = (\langle b_i, b_j \rangle)_{1 \leq i,j \leq \binom{2n}{n}}$. Its determinant D_n^B is called the Gram determinant of type B.

Type B Bilinear Form when $n = 4$

$\Big\langle$  ,  $\Big\rangle$ =  $= d^2 z$

The Gram Determinant of Type B when $n = 2$

When $n = 2$ there are six diagrams with crossingless connections:



The Gram matrix of Type B is

$$G_n^B = \begin{pmatrix} d^2 & dz & z^2 & z & d & z \\ dz & d^2 & dz & d & z & d \\ z^2 & dz & d^2 & z & d & z \\ z & d & z & d^2 & dz & z^2 \\ d & z & d & dz & d^2 & dz \\ z & d & z & z^2 & dz & d^2 \end{pmatrix}$$

and $D_n^B = (d^2 - z^2)^4(d^2 - 2 + z)(d^2 - 2 - z)$.

Theorem (Chen - Przytycki, Martin - Saleur)

Let $R = \mathbb{Z}[d, z]$*. Then,* $D_n^B(d, z) = \prod_{i=1}^{n} (T_i(d)^2 - z^2)^{\binom{2n}{n-i}}$*, where* T_i *denotes the Chebyshev polynomial of the first kind defined recursively by the equation* $T_{n+1}(d) = d \cdot T_n(d) - T_{n-1}(d)$*, with the initial conditions* $T_0(d) = 2$ *and* $T_1(d) = d$*.*

The Gram Determinant of Generalized Type A

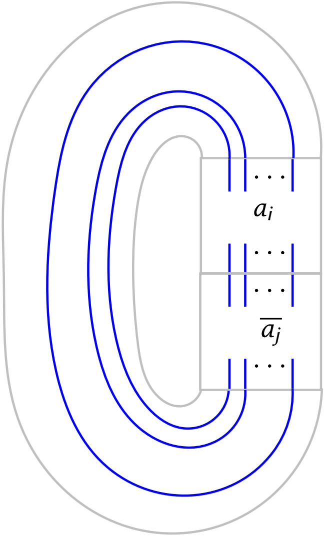
Let the disc D^2, with $2n$ marked points on its boundary, be considered as a rectangle with n points on the top edge and n points on the bottom edge. Define a bilinear form $\langle\ ,\ \rangle$ in the following way:

$$\langle\ ,\ \rangle : \mathcal{S}_{2,\infty}(D^2 \times I, \{x_i\}_1^{2n}) \times \mathcal{S}_{2,\infty}(D^2 \times I, \{x_i\}_1^{2n}) \longrightarrow \mathbb{Z}[d, z].$$

For $a_i, a_j \in \mathcal{A}_n$, glue a_i with the reflection, about the horizontal axis, of a_j which is denoted by $\bar{a}_j$, such that the bottom edge of a_i is identified with the top edge of $\bar{a}_j$. Connect the marked points on the top edge of a_i with those on the bottom edge of $\bar{a}_j$, in the annulus, respecting the ordering of the marked points.

Generalized Type A Bilinear Form

$\langle a_i, a_j \rangle =$



The Gram Determinant of Generalized Type A

The result is an annulus with two types of disjoint circles, homotopically trivial and non - trivial. Thus, we define, $\langle a_i, a_j \rangle = d^k z^m$ where k and m denote the number of these circles respectively.

Definition

We define the Gram matrix of generalized type A as $G_n^{A^{gen}} = (\langle a_i, a_j \rangle)_{1 \leq i,j \leq c_n}$, and denote its determinant by $D_n^{A^{gen}}$.

Theorem (B. - Ibarra - Mukherjee - Przytycki)

The Gram determinant of generalized type A is given by the following formula:

$$D_n^{A^{gen}}(d,z) = D_n^A(d) \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{S_{n-2i}(z)}{\Delta_{n-2i}(d)}\right)^{\left(\binom{n}{i}-\binom{n}{i-1}\right)^2},$$

where $S_k(z)$ denotes the Chebyshev polynomial of the second kind, defined recursively by the equation $S_{k+1}(z) = z \cdot S_k(z) - S_{k-1}(z)$, with the initial conditions $S_0(z) = 1$ and $S_1(z) = z$.

An Idempotent in the Temperley - Lieb Algebra

Definition (Jones)

Let $p : \mathbb{B}_n \longrightarrow S_n$ be a map, where $\mathbb{B}_n$ denotes the Artin braid group and S_n, the permutation group. This function sends a braid word to the induced permutation. Define $F_n = \sum_{\pi \in S_n} (A^3)^{|\pi|} b_\pi \in \mathbb{Z}[A^{\pm 1}]\mathbb{B}_n$, where $|\pi|$ is the length of the permutation π and b_π is the unique, minimal, positive braid such that $p(b_\pi) = \pi$.

The n^{th} Jones - Wenzl idempotent, denoted by f_n, is defined to be $\frac{F_n}{(\{n\}_{A^4})!} \in \mathbb{Q}(A)\mathbb{B}_n$, where $\{n\}_q = 1 + q + q^2 + \cdots + q^{n-1}$, and $(\{n\}_q)! = \{1\}_q \cdot \{2\}_q \cdots \{n\}_q$.[1]

- The Jones - Wenzl idempotent is evaluated in $TL_n(d)$ by taking the quotient of $\mathbb{Q}(A)\mathbb{B}_n$ by the Kauffman bracket skein relations.

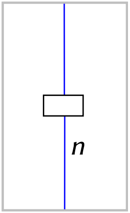
Jones - Wenzl Idempotents

Theorem (Wenzl)

The n^{th} Jones - Wenzl idempotent, $f_n \in TL_n(d)$, satisfies the following properties:

1. $f_n e_i = 0 = e_i f_n$, *for* $1 \leq i \leq n-1$,
2. $(f_n - \mathbb{1})$ *belongs to the algebra generated by* $\{e_1, e_2, \ldots, e_{n-1}\}$, *and*
3. $f_n f_n = f_n$.

$f_n =$



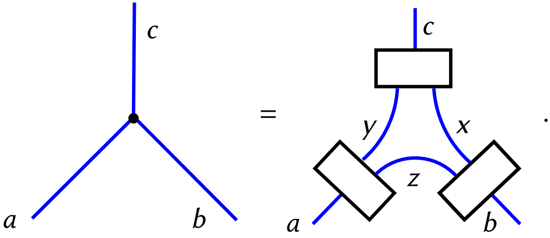
Admissible Points

Definition

The triple (a, b, c) of non - negative integers is said to be admissible if $a + b + c$ is even, $a \leq b + c$, $b \leq a + c$ and $c \leq a + b$.

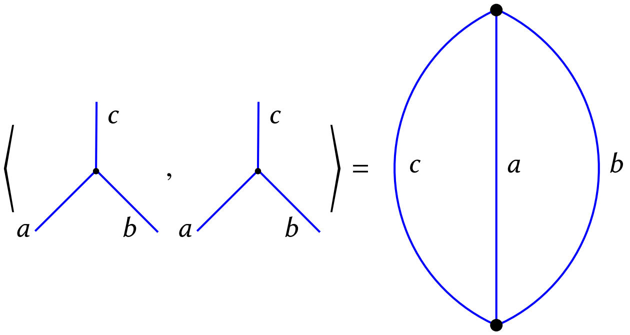
Definition

Let $a = y + z$, $b = x + z$, and $c = x + y$, that is, $x = \frac{(b+c-a)}{2}$, $y = \frac{(a+c-b)}{2}$, and $z = \frac{(a+b-c)}{2}$, then



Theta Nets

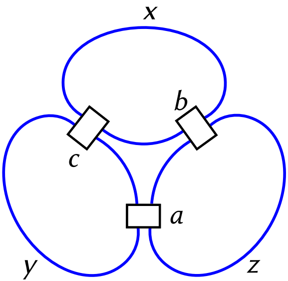
The following depicts the trace of the bilinear form of two triple point diagrams. The resulting picture is called a decorated Theta net.



Definition

A Theta net evaluation is the value of this net in the Temperley - Lieb algebra.

Theta Nets Continued

$\Theta(a, b, c) = \Gamma_{\mathbb{R}^2}(x, y, z) =$  .

Theorem (Masbaum - Vogel, Kauffman - Lins, Lickorish)

Let $\Delta_n!$ denote the product $\Delta_n \Delta_{n-1} \cdots \Delta_1$. Then

$$\Gamma_{\mathbb{R}^2}(x, y, z) = \frac{\Delta_{x+y+z}!\Delta_{x-1}!\Delta_{y-1}!\Delta_{z-1}!}{\Delta_{x+y-1}!\Delta_{x+z-1}!\Delta_{y+z-1}!}.$$

- $\Gamma_{Ann}(x, y, z) = \frac{S_{x+y}(x)}{\Delta_{x+y}} \Gamma_{\mathbb{R}^2}(x, y, z).$

Idea of the Proof: Change of Basis

- To compute the Gram determinant $D_n^{A^{gen}}$, we change the basis of the Temperley - Lieb alegbra so that the new basis elements are orthogonal to each other and in the new basis the Gram matrix is a diagonal matrix.
- This change of basis is given by an upper triangular matrix with 1's on the diagonal. Therefore, the Gram determinant is unchanged by the change of basis.

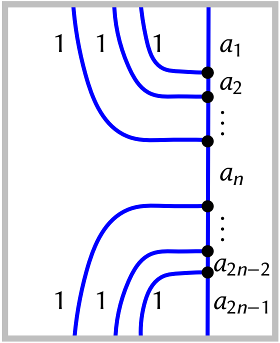
The Change of Basis

Definition

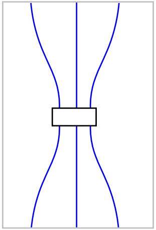
Consider the finite sequence $\{a_1, a_2, \ldots, a_{2n-1}\}$ of natural numbers which satisfies the following two conditions:

1. $a_1 = a_{2n-1} = 1$ and
2. $|a_i - a_{i-1}| = 1 \; \forall i$.

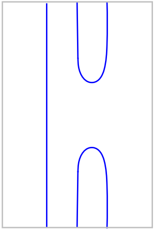
$D_{a_1, a_2, \ldots, a_{2n-1}} =$



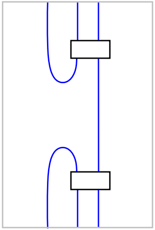
New Basis for $Tl_3(d)$



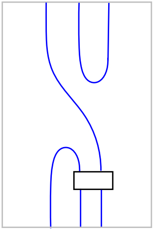
$D_{1,2,3,2,1}$



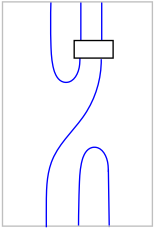
$D_{1,0,1,0,1}$



$D_{1,2,1,2,1}$



$D_{1,0,1,2,1}$



$D_{1,2,1,0,1}$

Idea of the Proof Continued

- To find the diagonal entries of the new Gram matrix, we use the properties of theta nets. The difference with the Gram determinant of generalized type A is that the theta nets are evaluated in the annulus and not in the disc and therefore, we deal with two-variable polynomials. Hence, we use the fact that the trace of the Jones - Wenzl idempotents, in the annulus, is the Chebyshev polynomial of the second kind.
- Finally, we use the combinatorial properties of mountain paths to find a closed formula for the Gram determinant of generalized type A.

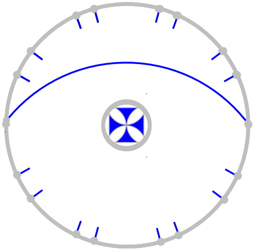
Crossingless Connections in the Möbius Band

Consider a Möbius band with $2n$ marked points on its boundary and let Mb_n denote the set of all diagrams of crossingless connections between these $2n$ points. Then the number of such diagrams is,

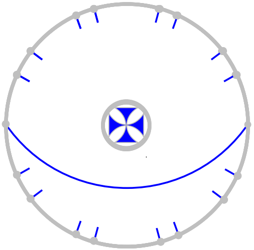
$$|Mb_n| = \sum_{k=0}^{n} \binom{2n}{k} = 2^{2n-1} + \frac{1}{2}\binom{2n}{n}.$$

In the formula $\sum_{k=0}^{n} \binom{2n}{k}$, k gives the number of curves that are disjoint from the crosscap.

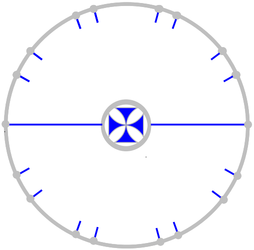
Crossingless Connection Diagrams when $n = 1$



b_1

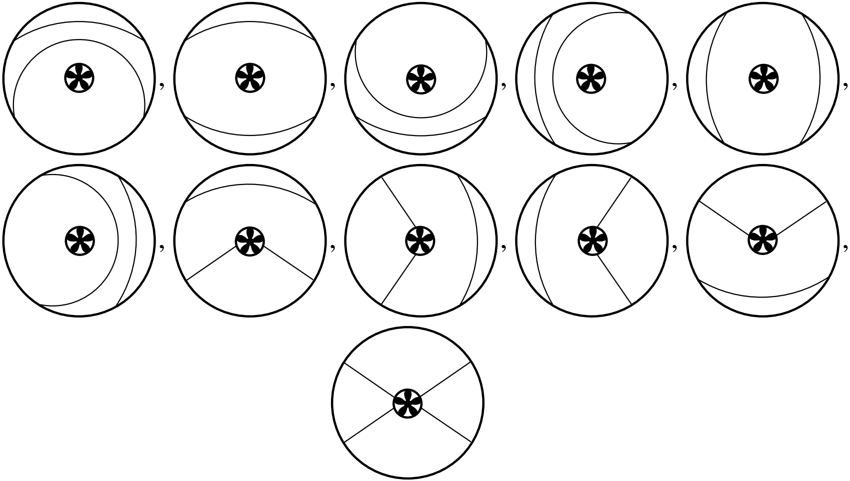


b_2



b_3

Crossingless Connection Diagrams when $n = 2$

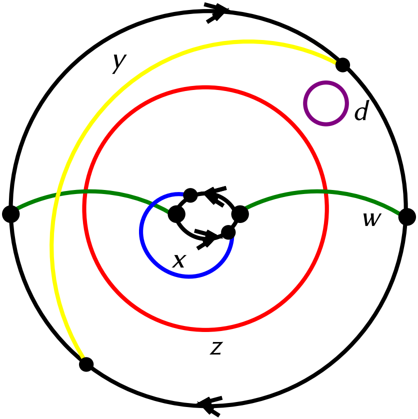


The Gram Determinant of Type Mb

Define a bilinear form $\langle\ ,\ \rangle$ in the following way:

$$\langle\ ,\ \rangle : \mathcal{S}_{2,\infty}(Mb \hat{\times} I, \{x_i\}_1^{2n}) \times \mathcal{S}_{2,\infty}(Mb \hat{\times} I, \{x_i\}_1^{2n}) \longrightarrow \mathbb{Z}[d, w, x, y, z],$$

where d, w, x, y, and z denote the five homotopically distinct simple closed curves on the Klein bottle.



The Gram Determinant of Type Mb

Given $m_i, m_j \in Mb_n$, glue the boundary component of m_i with that of the inversion of m_j along the marked circle, respecting the labels of the marked points. The result is a collection of disjoint simple closed curves on the Klein bottle which are of the five different types as shown before.

Thus, we define $\langle m_i, m_j \rangle = d^m x^n y^k z^l w^h$ where m, n, k, l and h denote the number of these simple closed curves, respectively.
The Gram matrix of type Mb is defined as
$G_n^{Mb} = (\langle m_i, m_j \rangle)_{1 \leq i,j \leq |Mb_n|}$ and its determinant is denoted by D_n^{Mb}.

The Gram Determinant of Type Mb when $n = 2$

$\langle\,,\,\rangle$											
	d^2	dz	z^2	z	d	z	dy	y	yz	y	z
	dz	d^2	dz	d	z	d	dy	y	dy	y	d
	z^2	dz	d^2	z	d	z	yz	y	dy	y	z
	z	d	z	d^2	dz	z^2	y	yz	y	dy	z
	d	z	d	dz	d^2	dz	y	dy	y	dy	d
	z	d	z	z^2	dz	d^2	y	dy	y	yz	z
	dx	dx	xz	x	x	x	dw	w	xy	w	x
	x	x	x	xz	dx	dx	w	dw	w	xy	x
	xz	dx	dx	x	x	x	xy	w	dw	w	x
	x	x	x	dx	dx	xz	w	xy	w	dw	x
	z	d	z	z	d	z	y	y	y	y	w^2

$$D_2^{Mb} = (d-z)^4[(d+z)w-2xy]^4(d^2(d^2-4))(d^2-2+z)[(d^2-2-z)(w^2-2)-2(2-z)]$$
$$= (T_1(d)-z)^4[(T_1(d)+z)T_1(w)-2xy]^4(T_4(d)-2)(T_2(d)+z)[(T_2(d)-z)T_2(w)-2(2-z)].$$

Chen's Conjecture

$$
\begin{aligned}
D_n^{(Mb)}(d,w,x,y,z) \quad = \quad & \prod_{k=1}^{n} (T_k(d) + (-1)^k z)^{\binom{2n}{n-k}} \\
& \prod_{\substack{k=1 \\ k \text{ odd}}}^{n} ((T_k(d) - (-1)^k z) T_k(w) - 2xy)^{\binom{2n}{n-k}} \\
& \prod_{\substack{k=1 \\ k \text{ even}}}^{n} ((T_k(d) - (-1)^k z) T_k(w) - 2(2-z))^{\binom{2n}{n-k}} \\
& \prod_{i=1}^{n} D_{n,i},
\end{aligned}
$$

where $D_{n,i} = \prod_{k=1+i}^{n} (T_{2k}(d) - 2)^{\binom{2n}{n-k}}$, and i represents the number of curves passing through the cross-cap.

Results [B. - Ibarra - Mukherjee - Przytycki]

Proposition

D_n^{Mb} is divisible by $(w(d+z) - 2xy)^{\binom{2n}{n-1}}$.

Theorem

$$\prod_{k=1}^{n} (T_k(d) + (-1)^k z)^{\binom{2n}{n-k}} \text{ divides } D_n^{Mb}.$$

Theorem

$$\prod_{k=1}^{n} (T_k(d) + (-1)^k z)^{\binom{2n}{n-k}} \text{ divides } D_n^{Mb}.$$

Thank you for viewing.